Alex M Mc Donald.

M S S M

A M Mc Donald

ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS

WITH

EXAMPLES AND APPLICATIONS

BY

JAMES M. TAYLOR, A.M., LL.D

PROFESSOR OF MATHEMATICS
COLGATE UNIVERSITY

REVISED EDITION

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON

COPYRIGHT, 1884, 1891
BY JAMES M. TAYLOR

COPYRIGHT, 1898
BY JAMES M. TAYLOR

ALL RIGHTS RESERVED
610.11

The Athenæum Press
GINN AND COMPANY · PRO-
PRIETORS · BOSTON · U.S.A.

PREFACE TO THE REVISED EDITION.

In this revision an attempt has been made to present in their unity the three methods commonly used in the Calculus. The concept of Rates is essential to a statement of the problems of the Calculus; the principles of Limits make possible general solutions of these problems, and the laws of Infinitesimals greatly abridge these solutions.

The Method of Rates, generalized and simplified, does not involve "the foreign element of time." For in measuring and comparing the rates of variables, the rate of any variable may be selected as the *unit of rates.* dy/dx is the x-rate of y, or the ratio of the rate of y to that of x, according as the rate of x is or is not the unit of rates.

The proofs of the principles of differentiation by the Method of Rates, and the numerous applications to geometry, mechanics, etc., found in Chapter II, render familiar the problem of rates before its solution by the Method of Limits or Infinitesimals is introduced.

In Chapter III, by proving that $\text{lt}\,(\Delta y/\Delta x) = dy/dx$, the problem of rates is reduced to the problem of finding the limit of the ratio of infinitesimals.

The Theory of Infinitesimals is that part of the Theory of Limits which treats of *variables having zero as their common limit.* In approaching its limit an infinitesimal passes through a series of finitely small values before it reaches infinitely small values. Infinitesimals can be divided into orders, and their laws can be established and applied when they are finitely small as well as when they are infinitely small. Hence, in the study of infinitesimals it is not necessary to determine that indefinite boundary between the finitely small and the infinitely small. Any small quantity *becomes* an infinitesimal when it *begins* to approach zero as its limit, not when it reaches any particular *degree* of *smallness*. A quantity, however small, which does not approach zero as its limit is not an infinitesimal. If it is recognized that the *essence* of infinitesimals lies in their *having zero as their limit*, rather than in their smallness, the study of them ceases to be mystical, obscure, and difficult.

Again, the concept of a limit as a constant whose value the variable never attains removes the necessity of studying the anatomy of Bishop Berkeley's "ghosts of departed quantities." Infinitesimals never equal zero and should not be denoted by the zero symbol. This distinction between infinitesimals and zero involves that between infinites and $a/0$.

The much-abused form $0/0$ cannot arise in the Calculus or elsewhere from any principle of limits; a distinctive service of the Theory of Limits is that it enables us to evaluate any determinate expression when it assumes this or any other indeterminate form.

Those who prefer to study the Calculus by the Method of Limits or Infinitesimals alone can omit the few demonstrations in Chapter II, which involve rates, and substitute for them the proofs by limits or infinitesimals in Chapter III.

To meet an increasing demand for a short course in differential equations, a chapter has been devoted to that subject.

A table of integrals arranged for convenience of reference is appended.

Throughout the work, as in previous editions, there are numerous practical problems from mechanics and other branches of applied mathematics which serve to exhibit the usefulness of the science, and to arouse and keep alive the interest of the student.

At the option of the teacher or reader, Chapters I and II of the Integral Calculus can be read after completing Chapters I and II of the Differential Calculus; also many of the numerous examples and problems may be omitted.

The author takes this opportunity of expressing his gratitude to the friends who by encouragement and suggestions have aided him in this revision.

JAMES M. TAYLOR.

HAMILTON, N. Y., 1898.

Digitized by the Internet Archive
in 2022 with funding from
Kahle/Austin Foundation

https://archive.org/details/elementsofdiffer0000jame

CONTENTS.

PART I. — DIFFERENTIAL CALCULUS.

CHAPTER I.

Functions. Rates. Differentials.

Sections | Pages
1–5. Classification of variables and functions 1, 2
6–8. Notation of functions. Continuity. Increments . . . 3–5
9, 10. Uniform change. Measure of rates 6, 7
11. Differentials 8
12, 13. dy/dx, a rate or a ratio of rates 9
14. Definition of the Differential Calculus 10

CHAPTER II.

Differentiation. Applications.

15–28. Differentiation of algebraic, logarithmic, and exponential functions 11–20
29, 30. Derivatives. Velocity 21–23
31–33. Tangent. $dy/dx =$ slope 24
34, 35. Equations of tangents and normals 25
36. Values of subt., subn., tan., and norm. 26–29
37–44. Differentiation of $\sin u$, $\cos u$, etc. 30–32
45–52. Differentiation of $\sin^{-1} u$, $\cos^{-1} u$, etc. 33–37
Miscellaneous examples 38, 39

CHAPTER III.

Problem of Rates Solved by Limits.

Sections — Pages

53–55. Limits. Notation. Lt $(\Delta y / \Delta x) = dy / dx$ 40–42
56, 57. Lt $(\Delta y / \Delta x)$, how found. Derivative as a limit 43
58–60. Infinitesimals. Infinites. Lt $(\Delta y / \Delta x) =$ slope . . . 43, 44
61, 62. Theorems concerning limits 45
63. Subt., subn., tan., and norm., in polar curves 46, 47
64–66. Orders of infinitesimals. Notation 48, 49
67–70. Theorems concerning infinitesimals 50
71. Rule for differentiating a function 51–53
72, 73. Orders of infinites. $\infty^n \cdot 0 \equiv 0$ 54
74. The symbol ∞ 55
75, 76. Use of infinitesimals. Limits in position 56

CHAPTER IV.

Successive Differentiation.

77. Successive differentials 57
78–80. Successive derivatives 58–61
81, 82. Leibnitz's theorem. Acceleration 62–64

CHAPTER V.

Indeterminate Forms.

83. Value of a function of x for $x = a$ 65
84. Indeterminate form $0/0$ 66
85. Indeterminate form ∞ / ∞ 67
86. Indeterminate forms $0 \cdot \infty$, $\infty - \infty$ 68
87. Indeterminate forms 0^0, ∞^0, $1^{\pm\infty}$ 69
88. Evaluation of derivatives of implicit functions 70

CHAPTER VI.

Expansion of Functions.

Sections Pages

89, 90. Series. Expansion of a function 71
91–93. Taylor's and Maclaurin's theorems 72–74
94. Expansions of $\sin x$ and $\cos x$ 75
95, 96. The exponential series. Second form of R_T . . . 76
97. The logarithmic series. 77
98. The binomial theorem 78
99. Failure of Taylor's and Maclaurin's theorems . . . 79–81

CHAPTER VII.

Maxima and Minima.

100. Definitions of maximum and minimum 82
101–104. Theorems concerning maxima and minima of fx . . 82, 83
105–107. Methods of examining fx for critical values of x . . 84, 87
Problems in maxima and minima 88–92

CHAPTER VIII.

Points of Inflexion. Curvature. Evolutes.

108–110. Points of inflexion 93, 94
111–113. Curvature 95
114. Circle of curvature 96
115. R in terms of x and y 96, 97
116. R in terms of ρ and θ 98
117. Co-ordinates of centre of curvature 98
118, 119. Properties of evolutes and involutes 99
120. Equations of evolutes 100–102

CHAPTER IX.

Envelopes. Order of Contact. Osculating Curves.

Sections Pages
121–123. Family of curves. Envelopes 103, 104
124. Equations of envelopes 104–106
125. Different orders of contact 107
126. Conditions of contact of the kth order 108
127–130. Osculating curves 109–111

CHAPTER X.

Change of the Independent Variable.

131, 132. Change of the independent variable 112–114

CHAPTER XI.

Functions of Two or More Variables.

133. Partial differentials and derivatives 115
134, 135. Total differentials 116
136. Derivative of an implicit function 117
137, 138. Derivative of a function of a function 118
139, 140. Successive partial differentials and derivatives . . 119–121
141. Successive total differentials 122
142. Expansion of $f(x+h, y+k)$ 122, 123
143, 144. Maxima and minima of $f(x, y)$ 123–126

CHAPTER XII.

Asymptotes. Singular Points. Curve Tracing.

145–148. Asymptotes to curves. *Rectangular axes* 127–131
149. Asymptotes to polar curves 132
150–153. Multiple points. Conjugate points 133–136
154, 155. Symmetry. Curve tracing 137–142

PART II. — INTEGRAL CALCULUS.

CHAPTER I.

Standard Forms. Direct Integration.

Sections Pages
156–158. General integrals. Elementary principles 143, 144
159–163. Standard formulas. Direct integration 145–157

CHAPTER II.

Definite Integrals. Applications.

164, 165. Definite and corrected integrals 158–160
166. Geometric meanings of $\int \phi(x)\,dx$, $\int_a^b \phi(x)\,dx$, and $\int_a^x \phi(x)\,dx$ 161, 162
167, 168. Accelerated motion. Change of limits 163–166

CHAPTER III.

Integration of Rational Fractions.

169–173. Decomposition and integration of fractions . . . 167–173

CHAPTER IV.

Integration by Rationalization.

174. Rationalization by substitution 174
175. Differentials involving $(a + bx)^{m/n}$ 174, 175
176, 177. Differentials involving $\sqrt{\pm x^2 + ax + b}$ 176, 177
178. Differentials of the form $x^m (a + bx^n)^{r/s} dx$ . . . 178–180

CHAPTER V.

Integration by Parts. Reduction Formulas.

179, 180. Integration by parts. Standard formulas . . . 181–184
181–183. Reduction formulas for $\int x^m (a + b\,x^n)^p dx$ . . . 185–189

CHAPTER VI.

Integration of Trigonometric Forms.

Sections | Pages

184–186. Forms, $\int \tan^n u\,du$, $\int \sec^n u\,du$, $\int \tan^m u \sec^n u\,du$. 190, 191

187–191. Form, $\int \sin^m u \cos^n u\,du$ 192–197

192. Forms, $\int \frac{du}{a+b\cos u}$, $\int \frac{du}{a+b\sin u}$ 198

193. Integration of trigonometric forms by substitution. 199, 200

194. $\int e^{ax} \sin bx\,dx$ 201

195. Integration by series 201, 202

CHAPTER VII.

Lengths and Areas of Curves. Surfaces and Volumes of Solids of Revolution.

196–200. Lengths of curves. Catenary. Tractrix 203–207

201, 202. Areas bounded by curves 208–210

203, 204. Surfaces and volumes of solids of revolution . . . 211–214

205. Volumes generated by plane figures 215, 216

CHAPTER VIII.

Double and Triple Integration. Applications.

206. Double and triple integration 217, 218

207, 208. Areas by double integration 219, 220

209. Surfaces by double integration 221, 222

210. Volumes of any solids by triple integration . . . 223, 224

211. Volumes of solids of revolution 225

212, 213. General formulas for centre of mass 226

214. Centre of mass of areas 227

215. Centre of mass of curves 228–230

216. Moment of inertia 231

CHAPTER IX.

Definite Integral as a Limit. Intrinsic Equations of Curves.

Sections Pages
217. Definite integral as a limit 232, 233
218–220. Intrinsic equations of curves 234–236

CHAPTER X.

Differential Equations.

221–223. Differential equations. General solutions . . . 237
224. Form, $\phi_1 xdx + \phi_2 ydy = 0$ 238, 239
225. Homogeneous equations 240
226. Non-homogeneous equations, linear in x and y . . 241, 242
227. Exact differential equations 243, 244
228. Integrating factor 245
229, 230. Rules for finding the integrating factor 246–248
231. Linear equations of the first order 249
232. Equations reducible to the linear form 250
233. Equations of the first order and nth degree . . . 251
234. Equations of orders above the first 252–257

APPENDIX.

Table of integrals 259–269

ELEMENTS OF THE CALCULUS.

PART I. DIFFERENTIAL CALCULUS.

CHAPTER I.

FUNCTIONS. RATES. DIFFERENTIALS.

1. A **variable** is a quantity which is, or is supposed to be, continually changing in value. Variable numbers are usually represented by the final letters of the alphabet, as x, y, z.

A **constant** is a quantity whose value is, or is supposed to be, fixed or invariable. Constant numbers may be *individual* or *general*. Individual constants are represented by figures; general constants are usually represented by the first letters of the alphabet.

For example, in the equation of the circle $(x-3)^2+(y-4)^2=r^2$, 3 and 4 are *individual* constants fixing the centre; r is a *general* constant denoting any radius; x and y are variables denoting the coördinates of the moving point which traces the circle.

2. Classification of variables. A variable whose value depends upon one or more other variables is called a *dependent variable*, or a *function* of those variables. A variable which does not depend upon any other variable is called an *independent variable.*

For example, x^3-b^2, $\sin x$, and $\log(x-a)$ are functions of the independent variable x. Again, $x^3+3xy+y^2$, $\log(x^2-y^2)$, and a^{x+y} are functions of the two variables x and y.

3. Classification of functions. An *algebraic* function is one which without the use of infinite series can be expressed by the operations of addition, subtraction, multiplication, division, and the operations denoted by constant exponents. All functions which are not algebraic are called *transcendental.* Of these, the more common are:

The *exponential* function $y = a^x$; and its inverse, the *logarithmic* function $x = \log_a y$.

The *trigonometric* functions $y = \sin x$, $y = \cos x$, etc.; and the *inverse-trigonometric* functions $x = \sin^{-1} y$, $x = \cos^{-1} y$, etc.

4. Explicit and implicit functions. When an equation involving two or more variables is solved for any one of them, this one is said to be an *explicit* function of the others. When an equation is not so solved, any one of its variables is called an *implicit* function of the others.

For example, in $x^2 + y^2 = r^2$, either y or x is an *implicit* function of the other; while in $y = \pm\sqrt{r^2 - x^2}$, y is an explicit function of x, and in $x = \pm\sqrt{r^2 - y^2}$, x is an explicit function of y.

A function is said to be one-valued, two-valued, or n-valued according as for each value of its variable it has one value, two values, or n values.

For example, y is a one-valued function in $y = x^3$, a two-valued function in $y^2 = 4px$, and a three-valued function in $y^3 + xy + x^2 = 1$.

5. Increasing and decreasing functions. An *increasing* function is one which *increases* when its variable increases. Hence, it decreases when its variable decreases.

A *decreasing* function is one which *decreases* when its variable *increases.* Hence, it increases when its variable decreases.

For example, $5x$ and $\log x$ are increasing functions of x, while $-5x$ and $1/x$ are decreasing functions of x.

6. Notation of functions. The symbols $f(x)$, $f'(x)$, $F(x)$, $\phi(x)$, and the like are used to denote different functions of x. Likewise, $f(x, y)$, $\phi(x, y)$ denote different functions of x and y.

The equation $y = f(x)$ expresses that y is an explicit function of x; while $f(x, y) = 0$ expresses that either y or x is an implicit function of the other.

In the same problem or discussion the symbol $f(\ )$ denotes the same function of one enclosed quantity as of another.

For example, if
$$f(x) \equiv x^2 + 2x + 3;$$
then in the same problem or discussion
$$f(y) \equiv y^2 + 2y + 3,$$
and
$$f(x + y) \equiv (x + y)^2 + 2(x + y) + 3.$$

Also, $f(2)$, $f(1)$, and $f(0)$ denote respectively the values of $f(x)$ for $x = 2$, $x = 1$, and $x = 0$.

EXAMPLES.

1. $f(x) \equiv 5x^2 - 3x + 2$; find $f(y)$, $f(x + h)$, $f(3)$, $f(0)$, $f(-2)$.

2. $f(\theta) \equiv \cos 2\theta$; find $f(x)$, $f(y)$, $f(0)$, $f(\pi/2)$, $f(\pi)$, $f(\theta + h)$.

3. $f(z) \equiv 2z^4 - z^3 + 1$, and $\phi(z) \equiv 7z^2 - 6z + 1$; show that
$$f(0) \equiv \phi(0), f(1) \equiv \phi(1), f(-2) \equiv \phi(-2).$$

4. $f(x) \equiv (a + x)^m$; find $f(z)$, $f(h - a)$, $f(0)$, $f(2)$, $f(x + h)$.

5. $F(x) \equiv e^x - e^{-x}$; show that $F(3x) \equiv [F(x)]^3 + 3F(x)$.

6. $f(x) \equiv \log \frac{1 - x}{1 + x}$; show that $f(x) + f(y) \equiv f\left(\frac{x + y}{1 + xy}\right)$.

7. $f(x + y) \equiv a^{x+y}$; find $f(x)$, $f(z)$, $f(m + n)$, $f(x + y + z)$.

8. $f(x, y) \equiv x^3 + 5xy^2 + 3y^3$; find $f(m, n)$, $f(2, -3)$, $f(-5, -7)$.

Find what $f(x + h) - f(x)$ and $f(y + k) - f(y)$ equal:

9. When $f(x) = 4x^2 + a$.

10. When $f(x) = 5x^3 + cx$.

11. When $f(x) = cx^4 - bx^2$.

12. When $f(x) = ax^4 + bx + c$.

7. A continuous real variable is a variable which in passing from one real value to another passes successively through all intermediate real values.

A function $f(x)$ is said to be real and continuous between $x = a$ and $x = b$, if when x is real and changes continuously from a to b, $f(x)$ is real and varies continuously from $f(a)$ to $f(b)$. In other words, $f(x)$ is real and continuous between $x = a$ and $x = b$ when the real locus of $y = f(x)$ between the points $[a, f(a)]$ and $[b, f(b)]$ is an unbroken curve.

Some functions are real and continuous for all real values of their variables; others are real and continuous only between certain limits.

For example, the time since any past event varies continuously. The velocity acquired by a falling body and the distance fallen are continuous functions of the time of falling. Most quantities in nature are continuous variables. Sin θ and cos θ are continuous functions for all real values of θ. Tan θ is a continuous function of θ between $\theta = 0$ and $\theta = \pi/2$, also between $\theta = \pi/2$ and $\theta = 3\pi/2$; but when θ passes through $\pi/2$ or $3\pi/2$, tan θ leaps from $+\infty$ to $-\infty$; hence, tan θ is *discontinuous* for $\theta = \pi/2$ or $3\pi/2$.

In $x^2 + y^2 = r^2$, y is a real and continuous function of x between $x = -r$ and $x = r$. In $a^2y^2 - b^2x^2 = -a^2b^2$, y is a real and continuous function of x between $x = -\infty$ and $x = -a$, and between $x = a$ and $x = \infty$. Between $x = -a$ and $x = a$, y is imaginary but continuous.

The Calculus treats of continuous variables only, or of variables between their limits of continuity.

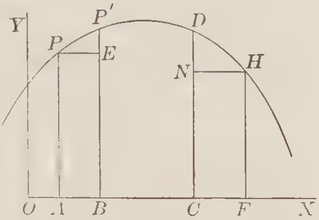
8. Increments. The amount of any change (increase or decrease) in the value of a variable is called an *increment*. If a variable is increasing, its increment is *positive*; if it is decreasing, its increment is *negative*.

An increment of a variable is denoted by writing the letter Δ before it; thus Δx, Δy, and $\Delta f(x)$ denote the increments of x, y, and $f(x)$, respectively.

If $y = f(x)$, Δx and Δy denote corresponding increments of x and y, and $\Delta y = \Delta f(x)$.

Let PH be the locus of $y = f(x)$ referred to the rectangular axes OX and OY.

If, when $x = OA$, $\Delta x = AB$,
then $\Delta y = BP' - AP = EP'$;
if, when $x = OC$, $\Delta x = CF$,
then $\Delta y = FH - CD = -ND$.



In the latter case Δy is negative, but it is properly called an *increment*, since it is what must be *added* to the first value CD to produce the second FH.

When $x = OA = x'$, $f(x) = AP = f(x')$;
when $x = OB = x' + \Delta x$, $f(x) = BP' = f(x' + \Delta x)$;
hence, if Δx is reckoned from the value x',

$$\Delta f(x) = BP' - AP = f(x' + \Delta x) - f(x').$$

This illustrates graphically the following general formula for $\Delta f(x)$:

$$\mathbf{\Delta f(x) \equiv f(x + \Delta x) - f(x).} \tag{a}$$

To prove (a) analytically, let x' denote any value of x; then
when $x = x'$, $f(x) = f(x')$;
when $x = x' + \Delta x$, $f(x) = f(x' + \Delta x)$.

Hence, while x changes from x' to $x' + \Delta x$, $f(x)$ changes from $f(x')$ to $f(x' + \Delta x)$; that is, if Δx is reckoned from the value x',

$$\Delta f(x) \equiv f(x' + \Delta x) - f(x'). \tag{1}$$

Since x' is any value of x, from (1) we have (a).

Ex. 1. The ratio of $\Delta(ax + b)$ to Δx is the constant a.

Let $f(x) \equiv ax + b$; then $f(x + \Delta x) \equiv a(x + \Delta x) + b$;
hence, by (a) we have

$$\Delta(ax + b) \equiv a(x + \Delta x) + b - (ax + b) \equiv a\Delta x. \quad \therefore \Delta(ax + b)/\Delta x \equiv a.$$

Ex. 2. The ratio of $\Delta(x^2)$ to Δx is the variable $2x + \Delta x$.

By (a), $\Delta(x^2) \equiv (x + \Delta x)^2 - x^2 \equiv 2x\Delta x + \Delta x^2$.

$$\therefore \Delta(x^2)/\Delta x \equiv 2x + \Delta x.$$

Ex. 3. Find $\Delta(ax^2 + bx)$; $\Delta(x^3)$; $\Delta(ax^3 - c)$; $\Delta(cx^4)$; $\Delta(cx^4 - bx^3)$.

By (a), $\Delta(ax^2 + bx) \equiv a(x + \Delta x)^2 + b(x + \Delta x) - (ax^2 + bx) \equiv \cdots$

9. Uniform and non-uniform change. When the increments of one of two variables are *proportional* to the corresponding increments of the other, either variable is said *to change uniformly with respect to the other.* When the corresponding increments of two variables are *not proportional,* either variable is said to change *non-uniformly* with respect to the other. E.g. the rectangle $ABCD$ in § 10 changes uniformly with respect to its variable base AB, while the triangle ABC changes non-uniformly with respect to its base.

Hence, $f(x)$ changes *uniformly* or *non-uniformly* with respect to x according as the ratio of $\Delta f(x)$ to Δx is *constant* or *variable.*

Ex. 1. With respect to x, does ax change uniformly or non-uniformly? $ax+b$? x^2? x^3? ax^2+bx? ax^3-c?

f(x) *changes uniformly with respect to* x *when, and only when,* f(x) *is a linear function of* x.

For if $f(x) = ax + b$, $\Delta f(x)/\Delta x = a$; hence, $f(x)$ changes uniformly with respect to x.

If $f(x)$ is not linear in x, $\Delta f(x)/\Delta x$ is variable; hence, $f(x)$ changes non-uniformly with respect to x.

Ex. 2. Show that y changes *uniformly* or *non-uniformly* with respect to x according as the point (x, y) traces a *straight* or a *curved* line.

10. Measure of rate. The *rate* of a variable is the rapidity of its change. Different variables have different rates, or change at different degrees of rapidity; and in general the same variable passes through different values at different rates.

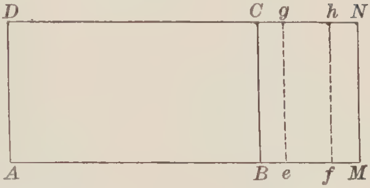
To measure the rates of variables, we fix upon some unit; that is, we select the rate of some variable as a *unit of rates.* For convenience, the rate of some *increasing* variable is always chosen as a unit of rates. If the rate of x is assumed as the unit of rates, we have the following definitions:

I. If a variable y changes uniformly with respect to x, *the measure of the rate of* y is the increment of y corresponding to the increment 1 of x.

By § 9, $ax + b$ changes uniformly with respect to x.

If $\Delta x = 1$, $\quad \Delta(ax + b) = a\Delta x = a$.

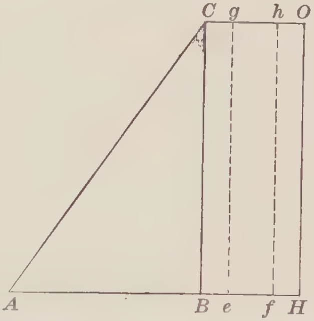
Hence, $ax + b$ changes uniformly at the rate of a to 1 of x.



Thus, $4x + 7$ changes uniformly at the rate of 4 to 1 of x; and $-8x + 9$ changes uniformly at the rate of -8 to 1 of x; that is, $-8x + 9$ *decreases* at the rate of 8 to 1 of x.

Again, if AD is constant, and AB is increasing, and BM equals a unit of the base, the rectangle $ABCD$ increases uniformly at the rate of $BMNC$ to a unit of the base.

II. If a variable changes non-uniformly with respect to x, *the measure of its rate* is what its increment corresponding to the increment 1 of x *would be* if at the value considered its change became uniform with respect to x.



Conceive a variable right triangle with the constant angle A as generated by the perpendicular moving to the right. The rate of the area of the triangle at the value ABC is evidently equal to the rate of the area of the rectangle $BCOH$. Hence, if BH equals a unit of the base, the rate of the area of the triangle at the value ABC is $BHOC$ to a unit of the base. Now, evidently, $BHOC$ is what would be the increment of the area corresponding to the increment BH, if at the value ABC the change of the area became uniform with respect to the base.

Suppose $BC = 2\,AB$, and let x and $2x$ denote the number of units in the base and perpendicular, respectively;

then $\quad$ area $ABC = x^2$ units,

and $\quad$ area $BHOC = 2x$ units, $\quad$ since $BH = 1$.

Hence, x^2 changes at the rate of $2x$ to 1 of x.

When $x = 2$, i.e. when x passes through the value 2, x^2 changes at the rate of 4 to 1 of x. When $x = 8$, x^2 changes at the rate of 16 to 1 of x.

When the rate of x is the unit of rates, the *measure of the rate* of a variable is called its **x-rate.**

The rate of a variable will be positive or negative according as the variable is increasing or decreasing; and conversely.

11. Differentials. The *differentials* of variables which change uniformly with respect to the same variable are their corresponding increments.

The *differentials* of variables which change non-uniformly are what *would be* their corresponding increments if at the corresponding values considered the change of each became and continued uniform with respect to the same variable. Hence the differential of a variable will be positive or negative according as the variable is increasing or decreasing.

The differential of a variable is denoted by writing the letter d before it; thus, dx read "differential x" is the symbol for the differential of x. When the symbol of a function is not a single letter parentheses are used; thus, $d(x^3)$ and $d(x^2 - 2x)$ denote the differentials of x^3 and $x^2 - 2x$, respectively.

In the first figure of § 10,

$$\text{if } Be = d\,(\text{base}), \qquad BegC = d\,(\text{rectangle});$$

$$\text{if } Bf = d\,(\text{base}), \qquad BfhC = d\,(\text{rectangle}).$$

In the second figure of § 10,

$$\text{if } Be = d\,(\text{base}), \qquad BegC = d\,(\text{triangle});$$

$$\text{if } Bf = d\,(\text{base}) = dx, \quad BfhC = d\,(\text{triangle}) = d\,(x^2).$$

$$\therefore \frac{d\,(x^2)}{dx} = \frac{BC \cdot Bf}{Bf} = BC = 2x = \text{the } x\text{-rate of } x^2, \qquad \text{§ 10}$$

which is a particular case of the next theorem.

12. dy/dx, a rate or a ratio of rates.

I. *When the rate of* x *is the unit of rates.*

In this case dx is positive. Let n be so chosen that ndx will be equal to 1; then ndy will denote what would be the increment of y corresponding to the increment 1 of x, if at the value considered the change of y became uniform with respect to x; hence,

$$ndy = \text{the } x\text{-rate of } y. \qquad \S\, 10$$

$$\therefore \frac{dy}{dx} = \frac{ndy}{ndx} = ndy = \text{the } x\text{-rate of } y.$$

That is, *the differential of one variable divided by that of another whose rate is the unit of rates is equal to the rate of the first variable.*

II. *When the rate of* x *is not the unit of rates.*

Let the rate of v be the unit of rates; then

$$\frac{dy}{dx} = \frac{dy/dv}{dx/dv} = \frac{\text{the } v\text{-rate of } y}{\text{the } v\text{-rate of } x}. \tag{1}$$

That is, *the ratio of the differentials of any two variables is equal to the ratio of their rates.*

In this case dx may be either positive or negative.

COR. 1. If $y = f(x)$, y is an increasing or a decreasing function of x according as dy/dx is positive or negative. (Why?)

In practical affairs and physical science the more common unit of rates is the rate of time. Thus, we speak of a distance as increasing at the rate of 20 miles an hour, meaning thereby that if at the value considered the change of this distance became uniform with respect to time, its increment in an hour *would be* 20 miles.

COR. 2. If t denotes the number of units in a portion of time, then dy/dt, or the t-rate of y, gives the time-rate of y.

13. The **meanings** of Δy, dy, and the x-rate of y should be carefully considered. Δy denotes the *actual* increment of y corresponding to Δx. dy denotes what the increment of y corresponding to the increment dx *would be*, if at the value considered the change of y became uniform with respect to x.

Hence, $dy = \Delta y$ only when y changes uniformly with respect to x and $dx = \Delta x$.

The x-rate of y, which equals dy/dx, is what the increment of y corresponding to the increment 1 of x *would be* if at the value considered the change of y became uniform with respect to x. When the rate of x is not the unit of rates dy/dx means simply the ratio of the rate of y to that of x.

14. The **problem** of the **Differential Calculus** is to measure and compare the rates of change of continuous variables when the relation of the variables is known or given.

The inverse problem of finding the relation of the variables themselves when their relative rates are known is the *problem* of the **Integral Calculus.**

EXAMPLES.

1. With respect to v, does av change uniformly or non-uniformly? $av + c$? v^2? av^2? $v^2 + av$? $\sin v$?

2. What is the t-rate of at? $at + b$? t^2? $t^2 + at$? $t^2 + bt + c$?

3. What is the v-rate of av? $av + c$? v^2? $v^2 + av + c$?

4. Conceiving a square as generated by two of its sides, illustrate that $d(x^2) = 2x dx$, and that x^2 changes at the rate of $2x$ to 1 of x.

5. Conceiving a cube as generated by three of its faces, illustrate that $d(x^3) = 3x^2 dx$, and that x^3 changes at the rate of $3x^2$ to 1 of x.

6. Conceiving the point (x, y) as tracing any curve, draw the lines which represent dy and dx at any point P, and show that dy/dx is the slope of the tangent to the curve at P. (See § 33.)

7. For what real values of x is $\sqrt{a^2 - x^2}$ real and continuous? $\sqrt{x^2 - a^2}$? $\sqrt{(a+b)^2 - x^2}$? a/x? $a/(c - x)$? $\cot x$? $\log x$?

8. If $f(x, y) \equiv e^x - e^{-y}$, prove

$$f(3x, 3y) \equiv [f(x, y)]^3 + 3e^x e^{-y} f(x, y).$$

9. If $f(\theta) \equiv \dfrac{\theta - 1}{\theta + 1}$, show that $\dfrac{f(\theta) - f(x)}{1 + f(\theta) \cdot f(x)} \equiv \dfrac{\theta - x}{1 + \theta x}$.

CHAPTER II.

DIFFERENTIATION. APPLICATIONS.

15. Differentiation is the operation of finding the differential of a function. The sign of differentiation is the letter d; thus d in the expression $d(x^3)$ indicates the operation of differentiating x^3, while the whole expression $d(x^3)$ denotes the differential of x^3.

In the *following formulas*, u, y, v, w, and z denote different functions of x. The rate of x will be used as the unit of rates.

16. If $\mathbf{u = y}$, $\mathbf{du = dy}$. [1]

That is, *the differentials of equals are equal.*

For if u continually equals y, it is self-evident that u and y must change at equal rates; that is,

$$du/dx = dy/dx; \quad \therefore du = dy.$$

For example, if $y^2 = 4ax$, $d(y^2) = d(4ax)$.

17. $\mathbf{d(a) \equiv o}$. [2]

That is, *the differential of a constant is zero.*

For the rate of any constant a is zero; that is,

$$da/dx = 0; \quad \therefore da = 0.$$

18. $\mathbf{d(u + y + \cdots + z + a) \equiv du + dy + \cdots + dz}$. [3]

That is, *the differential of a polynomial is the sum of the differentials of its terms.*

For it is self-evident that the rate of the sum $u + y + \cdots + z + a$ is equal to the sum of the rates of its parts $u, y, \cdots, z$, and a; that is,

$$\frac{d(u + y + \cdots + z + a)}{dx} \equiv \frac{du}{dx} + \frac{dy}{dx} + \cdots + \frac{dz}{dx} + \frac{da}{dx}.$$

Multiplying by dx, since $da = 0$, we obtain [3].

For example, $d(3x^2 - 4x^2 - 2) = d(3x^2) + d(-4x^2) + d(-2)$.

19. $\mathbf{d(au) \equiv adu.}$ [4]

That is, *the differential of the product of a constant and a variable is the product of the constant and the differential of the variable.*

By § 9, au changes uniformly with respect to u.

By § 8, $\Delta(au) = a\Delta u.$

Hence, by the definition in § 11, we obtain [4].

For example, $d(3ax^3) = 3a \cdot d(x^3)$, and $d\left(\frac{z}{a}\right) = d\left(\frac{1}{a} \cdot z\right) = \frac{1}{a}dz.$

20. $\mathbf{d(\log_a u) \equiv m \cdot du / u}$, *where* **u** *is positive.* [5]

That is, *the differential of the logarithm of a variable is the modulus of the system into the differential of the variable divided by the variable.*

For, n being a general constant, let

$$u = ny. \tag{1}$$

$$\therefore \log_a u = \log_a n + \log_a y. \tag{2}$$

From (2), by [1], [2], and [3] we obtain

$$d(\log_a u) = d(\log_a y). \tag{3}$$

Differentiating (1) and dividing by (1), we obtain

$$du / u = dy / y. \tag{4}$$

Dividing (3) by (4), we obtain

$$d(\log_a u) : du / u = d(\log_a y) : dy / y. \qquad (5)$$

It remains to prove that the equal ratios in (5) are constant.

Let m denote the common ratio in (5) when $y = y'$; then

$$d(\log_a u) = m \cdot du / u, \qquad (6)$$

when $u = ny'$. But, as n is a general constant, ny' denotes any number; hence (6), or [5], holds true for all values of u, m being a constant.

The constant m is called the **modulus** of the system of logarithms whose base is a.

The modulus of the common system of logarithms, obtained in § 97, is $0.434294 \cdots$.

From the nature of logarithms we know that $\log_a u$ changes the faster the smaller we take a. From [5] we learn that $\log_a u$ changes at the rate of m/u to 1 of u. Hence, the modulus m varies with the base a.

Ex. The number u changes how many times as fast as $\log_{10} u$ when $u = 2560$?

$$\frac{du}{d(\log_a u)} = \frac{u}{m} = \frac{2560}{0.43429 \cdots} = 5895, \text{ nearly.}$$

That is, u changes nearly 5895 times as fast as its common logarithm when $u = 2560$.

21. Natural logarithms.

The system of logarithms whose modulus is unity is called the *Naperian* or *natural* system. The symbol for the base of the natural system is e.

Hence, $$\mathbf{d(\log_e u) \equiv du / u.} \qquad [6]$$

Natural logarithms are evidently the simplest and most natural for analytic purposes.

Hereafter when no base is written, e is understood.

In the natural system $du = u \cdot d(\log u)$; that is, the number u changes u times as fast as its natural logarithm.

22. $\mathbf{d(uy) \equiv ydu + udy.}$ [7]

When u and y are both positive, we have

$$\log(uy) = \log u + \log y;$$

$$\therefore \frac{d(uy)}{uy} = \frac{du}{u} + \frac{dy}{y}; \qquad \text{by [6]}$$

$$\therefore d(uy) = ydu + udy. \qquad (1)$$

Multiplying (1) by ab, by [4] we obtain

$$d(au \cdot by) = by \cdot d(au) + au \cdot d(by). \qquad (2)$$

Since a and b are general constants, au and by denote any variables, real or imaginary. Hence [7] holds true for any real or imaginary values of u and y.

23. $\mathbf{d(uvyz \cdots)}$
$\mathbf{\equiv (vyz \cdots)du + (uyz \cdots)dv + (uvz \cdots)dy + \cdots}$. [8]

That is, *the differential of the product of any number of variables is the sum of the products of the differential of each into all the rest.*

If in [7] we put vw for u, we obtain

$$d(vwy) = yd(vw) + vwdy$$
$$= wydv + vydw + vwdy. \qquad (1)$$

By repeating this process the theorem is proved for any number of variables.

If $v = w = y$, (1) becomes $d(y^3) = 3y^2dy$.

24. $\mathbf{d(u/y) \equiv (ydu - udy)/y^2}$. [9]

That is, *the differential of a fraction is the denominator into the differential of the numerator minus the numerator into the differential of the denominator, divided by the square of the denominator.*

Let $z = u/y$; then $zy = u$.

$$\therefore ydz + zdy = du; \qquad \text{by [1], [7]}$$

$$\therefore dz = (du - zdy)/y,$$

or
$$d\left(\frac{u}{y}\right) = \frac{du - (u/y)dy}{y} = \frac{ydu - udy}{y^2}.$$

COR. $\mathbf{d(a/y) \equiv -ady/y^2}$. [10]

For
$$d\frac{a}{y} = \frac{yda - ady}{y^2} = -\frac{ady}{y^2}, \text{ since } da = 0.$$

25. Differentials of $\mathbf{u^y}$, $\mathbf{b^y}$, $\mathbf{u^n}$. When u is positive,

$$\log_a(u^y) = y \log_a u;$$

$$\therefore m\frac{d(u^y)}{u^y} = my\frac{du}{u} + \log_a u \cdot dy; \qquad \text{by [5]}$$

$$\therefore \mathbf{d(u^y) \equiv yu^{y-1}du + u^y \log_a u\, dy/m}. \qquad [11]$$

Putting b for u in [11], by [2] we obtain

$$\mathbf{d(b^y) \equiv b^y \log_a b\, dy/m}. \qquad [12]$$

Putting n for y in [11], by [2] we obtain

$$\mathbf{d(u^n) \equiv n\, u^{n-1}du}. \qquad [13]$$

Multiplying [13] by c^n, by [4] we obtain

$$d(cu)^n = n(cu)^{n-1}d(cu). \qquad (1)$$

Since cu in (1) denotes any variable base, [13] holds true for any value of u, positive or negative, real or imaginary. The function u^y is continuous only when u is positive; hence [11] and [12] are limited to positive values of u and b.

26. Stating [13] in words, we have

The differential of a variable base affected with a constant exponent is the product of the exponent, the base with its exponent diminished by one, and the differential of the base.

COR. $\mathbf{d\sqrt{u} \equiv du/2\sqrt{u}}$. [14]

For $d(u^{1/2}) = \frac{1}{2} u^{-1/2} du = du/2\sqrt{u}$.

27. Stating [12] in words, we have

The differential of an exponential function with a positive constant base is the function itself into the logarithm of the base into the differential of the exponent, divided by the modulus of the system of logarithms used.

COR. In the natural system, $m = 1$; hence, as $\log e = 1$, from [12] we have

$$\mathbf{d(b^y) \equiv b^y \log b\, dy;} \qquad [15]$$

and $$\mathbf{d(e^y) \equiv e^y dy.} \qquad [16]$$

28. Comparing [13] and [12] with [11], we see that

The differential of an exponential function with a positive variable base can be obtained by first differentiating as though the exponent were constant, and then as though the base were constant.

EXAMPLES.

By one or more of the preceding formulas exclusive of [5], [6], [11], [12], [15], [16], differentiate

1. $y = x^3 - 8x + 2x^2$.

$$dy = d(x^3 - 8x + 2x^2) \qquad \text{by [1]}$$
$$= d(x^3) + d(-8x) + d(2x^2) \qquad \text{by [3]}$$
$$= 3x^2dx - 8dx + 4xdx. \qquad \text{by [4], [13]}$$
$$\therefore dy/dx = 3x^2 - 8 + 4x = \text{the } x\text{-rate of } y. \qquad \S\ 12$$

That is, y, or $x^3 - 8x + 2x^2$, changes at the rate of $3x^2 - 8 + 4x$ to 1 of x; or y changes $3x^2 - 8 + 4x$ times as fast as x.

When $x = -4$, y is *increasing* at the rate of 24 to 1 of x;
when $x = 0$, y is *decreasing* at the rate of 8 to 1 of x;
when $x = 5$, y is *increasing* at the rate of 87 to 1 of x.

2. $y = 3ax^2 - 5nx - 8m$. $\qquad dy = (6ax - 5n)\,dx$.

Note. At first the student should give the meaning of each differential equation which he obtains.

3. $y = 5ax^2 - 3b^2x^3 - abx^4$. $\qquad dy = (10ax - 9b^2x^2 - 4abx^3)\,dx$.

4. $y = a^3 + 5b^2x^3 + 7a^3x^5$. $\qquad dy = (15b^2x^2 + 35a^3x^4)\,dx$.

✓ 5. $y = ax^{3/2} + bx^{1/2} + c$. $\qquad dy/dx = (3ax + b)/2\sqrt{x}$.

6. $y = (b + ax^2)^{5/4}$. $\qquad dy = \frac{5}{2}(b + ax^2)^{1/4}\,axdx$.

7. $y = (cx + bx^3)^{4/3}$. $\qquad dy = \frac{4}{3}(cx + bx^3)^{1/3}(c + 3bx^2)\,dx$.

8. $y = (1 + 2x^2)(1 + 4x^3)$. $\qquad dy = 4x(1 + 3x + 10x^3)\,dx$.

$$dy = (1 + 2x^2)\,d(1 + 4x^3) + (1 + 4x^3)\,d(1 + 2x^2).$$

9. $y = (x + 1)^5(2x - 1)^3$. $\qquad dy = (16x + 1)(x + 1)^4(2x - 1)^2\,dx$.

10. $y = (a + x)\sqrt{a - x}$. $\qquad \dfrac{dy}{dx} = \dfrac{a - 3x}{2\sqrt{a - x}}$.

11. $y = (x^2 + 1)^2(2x^2 + x)^3$.

$$dy = (x^2 + 1)(2x^2 + x)^2(20x^3 + 7x^2 + 12x + 3)\,dx.$$

12. $y = (1 - 3x^2 + 6x^4)(1 + x^2)^3$. $\qquad dy = 60x^5(1 + x^2)^2\,dx$.

13. $y = x^6(a + 2x)^3(a - 3x)^2$.

$$dy = 6x^5(a + 2x)^2(a - 3x)(a^2 - ax - 11x^2)\,dx.$$

14. $y = \dfrac{x + a^2}{x + b}$. $\qquad \dfrac{dy}{dx} = \dfrac{b - a^2}{(x + b)^2}$.

$$dy = \frac{(x + b)\,d(x + a^2) - (x + a^2)\,d(x + b)}{(x + b)^2}.$$

15. $y = \dfrac{2x^4}{a^2 - x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{8a^2x^3 - 4x^5}{(a^2 - x^2)^2}$.

16. $y = \sqrt{ax^2 + bx + c}$. $\qquad \dfrac{dy}{dx} = \dfrac{2ax + b}{2\sqrt{ax^2 + bx + c}}$.

17. $y = \dfrac{2x^2 - 3}{4x + x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{8x^2 + 6x + 12}{(4x + x^2)^2}$.

18. $y = \sqrt{\dfrac{1 + x}{1 - x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1 - x)\sqrt{1 - x^2}}$.

19. $y = \dfrac{x^3}{(1 + x)^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{3x^2 + x^3}{(1 + x)^3}$.

20. $y = \dfrac{2x^2 - 1}{x\sqrt{1 + x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1 + 4x^2}{x^2(1 + x^2)^{3/2}}$.

21. $y = \dfrac{a^2 - b^2}{(2ax - x^2)^{3/2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{3(a^2 - b^2)(x - a)}{(2ax - x^2)^{5/2}}$.

22. $y = \sqrt{ax} + \sqrt{c^2x^3}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a} + 3cx}{2\sqrt{x}}$.

23. $y = \dfrac{x^n + 1}{x^n - 1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2nx^{n-1}}{(x^n - 1)^2}$.

24. $y = \dfrac{bx}{\sqrt{2ax - x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{abx}{(2ax - x^2)^{3/2}}$.

25. $y = \dfrac{x^n}{(1 + x)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{nx^{n-1}}{(1 + x)^{n+1}}$.

26. $y = \dfrac{1}{(a + x)^m(b + x)^n}$. $\qquad \dfrac{dy}{dx} = -\dfrac{m(b + x) + n(a + x)}{(a + x)^{m+1}(b + x)^{n+1}}$.

27. $y = (a + x)^m(b + x)^n$.

$$dy = \{m(b + x) + n(a + x)\}(a + x)^{m-1}(b + x)^{n-1}dx.$$

28. $y = \dfrac{x^{2n}}{(1 + x^2)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{2nx^{2n-1}}{(1 + x^2)^{n+1}}$.

29. $y = x^{12}(b - 3x)^4(c + 4x)^3$.

$$dy = 12x^{11}(b - 3x)^3(c + 4x)^2(bc - 4cx + 5bx - 19x^2)\,dx.$$

30. $y = \dfrac{1 - x}{\sqrt{1 + x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{-(1 + x)}{(1 + x^2)^{3/2}}$.

31. $y = \dfrac{\sqrt{x^2 - a^2}}{x}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{x^2\sqrt{x^2 - a^2}}$.

32. $y = \dfrac{x}{\sqrt{a^2 + x^2} - x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{a^2}\left\{\dfrac{a^2 + 2x^2}{\sqrt{a^2 + x^2}} + 2x\right\}$.

Rationalize the denominator before differentiating.

33. $y = \dfrac{x^3}{x + \sqrt{1 + x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4x^4 + 3x^2}{\sqrt{x^2 + 1}} - 4x^3$.

34. $y = \dfrac{x}{\sqrt{x^2 + a^2} - a}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a}{x^2}\left\{1 + \dfrac{a}{\sqrt{x^2 + a^2}}\right\}$.

EXAMPLES.

By one or more of the sixteen preceding formulas, differentiate

1. $y = \log(x^2 + x)$. $\qquad \dfrac{dy}{dx} = \dfrac{2x + 1}{x^2 + x}$.

2. $y = \log_a x^3 = 3\log_a x$.

3. $y = \log_a \sqrt{1 - x^3} = \frac{1}{2}\log_a(1 - x^3)$. $\qquad \dfrac{dy}{dx} = \dfrac{3mx^2}{2(x^3 - 1)}$.

4. $y = x \log x$. $\qquad dy = (\log x + 1)\,dx$.

5. $y = \log \dfrac{1 + \sqrt{x}}{1 - \sqrt{x}} = \log(1 + \sqrt{x}) - \log(1 - \sqrt{x})$.

6. $y = \log\{\sqrt{1 - x}\,(1 + x)\}$. $\qquad \dfrac{dy}{dx} = \dfrac{1 - 3x}{2(1 - x^2)}$.

7. $y = \log(1 + x^2)^4$. $\qquad \dfrac{dy}{dx} = \dfrac{8x}{1 + x^2}$.

8. $y = \log(\sqrt{1 + x^2} + x)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + x^2}}$.

9. $y = (\log x)^3$. $\qquad$ **10.** $y = c^{\log ax}$.

11. $y = x^x$. $\qquad dy = x^x(\log x + 1)\,dx$.

12. $y = x^{x^x}$. $\qquad \dfrac{dy}{dx} = x^{x^x} x^x\left(\log x(\log x + 1) + \dfrac{1}{x}\right)$.

Here $\log y = x^x \log x$; $\therefore \dfrac{dy}{y} = x^x \dfrac{dx}{x} + \log x\,[x^x(\log x + 1)]\,dx$.

13. $y = x^{e^x}$. $\qquad \dfrac{dy}{dx} = x^{e^x} e^x \dfrac{1 + x\log x}{x}$.

14. $y = e^{x^x}$. $\qquad dy = e^{x^x} x^x (\log x + 1)\, dx.$

15. $y = x^{\log x}$. $\qquad dy = 2\, x^{\log x - 1} \log x \cdot dx.$

16. $y = \dfrac{a^x - 1}{a^x + 1}$. $\qquad \dfrac{dy}{dx} = \dfrac{2\, a^x \log a}{(a^x + 1)^2}$.

17. $y = \log \dfrac{x}{\sqrt{1 + x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\, (1 + x^2)}$.

18. $y = \log (\log x)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x \log x}$.

19. $y = \dfrac{c^x}{x^x}$. $\qquad \dfrac{dy}{dx} = \left(\dfrac{c}{x}\right)^x \left(\log \dfrac{c}{x} - 1\right)$.

20. $y = \sqrt{\dfrac{1 - x}{x^2 + x + 1}}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2 - 2\,x - 2}{2 \sqrt{1 - x}\, (x^2 + x + 1)^{3/2}}$.

In example 20 and some that follow, pass to logarithms.

21. $y = \dfrac{x^n}{(a + x)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{nax^{n-1}}{(a + x)^{n+1}}$.

22. $y = \dfrac{(a^2 + x^2)^{3/2}}{(a^2 - x^2)^{1/2}}$. $\qquad \dfrac{dy}{dx} = 2\, x \dfrac{2\, a^2 - x^2}{(a^2 - x^2)^{3/2}} (a^2 + x^2)^{1/2}$.

23. $y = \log \dfrac{e^x}{1 + e^x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1 + e^x}$.

24. $y = x^{1/x}$. $\qquad \dfrac{dy}{dx} = \dfrac{(1 - \log x)\, x^{1/x}}{x^2}$.

25. $y = e^x (1 - x^3)$. $\qquad dy = e^x (1 - 3\, x^2 - x^3)\, dx.$

26. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{(e^x + e^{-x})^2}$.

27. $y = \left(\dfrac{x}{n}\right)^{nx}$. $\qquad \dfrac{dy}{dx} = n \left(\dfrac{x}{n}\right)^{nx} \left(1 + \log \dfrac{x}{n}\right)$.

28. $y = (a^x + 1)^2$. $\qquad dy = 2\, a^x (a^x + 1) \log a\, dx.$

29. $y = a^{c^x}$. $\qquad$ 30. $y = \dfrac{x}{e^x - 1}$.

29. Derivatives. The ratio of the differential of a function of a single variable to that of the variable is called the *derivative* of the function. Thus, the derivative of y as a function of x, i.e. dy/dx, is the *x-rate* of y or *a ratio of rates.*

The derivative of $f(x)$ is denoted by $f'(x)$.

That is, $$df(x)/dx \equiv f'(x).$$

The operation of finding the derivative of a function of x is denoted by $\frac{d}{dx}$. Thus $\frac{d}{dx}(3x^5) \equiv \frac{d(3x^5)}{dx} \equiv 15x^4$.

A derivative is often called a *differential coefficient.*

EXAMPLES.

In each of the following equations find the derivative of the implicit function y:

1. $x^3 + y^3 = 3axy + c$.

$$d(x^3 + y^3) = d(3axy + c); \qquad \text{by [1]}$$
$$\therefore 3x^2dx + 3y^2dy = 3aydx + 3axdy;$$
$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

2. $y^3 = 2x^2y + bx$. $\qquad \frac{dy}{dx} = \frac{4xy + b}{3y^2 - 2x^2}$.

3. $y^2 = 4px$. $\qquad$ 4. $a^2y^2 + b^2x^2 = a^2b^2$.

5. $x^3 + 3axy = -y^3$. $\qquad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$.

6. $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$. $\qquad \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{m-1}\left(\frac{b}{a}\right)^m$.

7. $e^{x+y} = xy$. $\qquad \frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$.

Passing to logarithms, we have $x + y = \log x + \log y$.

8. $x^y = y^x$. $\qquad \frac{dy}{dx} = \frac{y^2 - xy\log y}{x^2 - xy\log x} = \left(\frac{y}{x}\right)^2 \frac{\log x - 1}{\log y - 1}$.

9. $x^{y+x} = y^{y-x}$. $$\frac{dy}{dx} = \frac{xy \log (xy) + y (y + x)}{xy \log (y / x) + x (y - x)}.$$

10. $(y + x)^x = x^x a^y$. $$\frac{dy}{dx} = \frac{y + (x + y) [\log x - \log (x + y)]}{x - (x + y) \log a}.$$

30. The **velocity** of a moving body is the time-rate of the distance traversed by it. Let s denote the distance, t the time, and v the velocity; then $v =$ the time-rate of $s = ds / dt$.

EXAMPLES.

1. The area of a circular plate of metal is expanding by heat. When the radius passes through the value 2 in. it is increasing at the rate of 0.01 in. a second; how fast is the area increasing?

Let $x =$ the number of in. in the radius,
and $y =$ the number of sq. in. in the area.
Then $y = \pi x^2$; $\therefore dy / dt = 2 \pi x \cdot dx / dt$. (1)
When $x = 2$, $dx / dt = 0.01$; $\therefore dy / dt = 0.04 \pi$.

That is, the area is increasing at the rate of 0.04π sq. in. a second when it passes through the value considered.

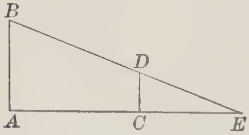
2. When the radius of a spherical soap-bubble equals 3 in. it is increasing at the rate of 2 in. a second; how fast is its volume increasing?

Ans. 72π cu. in. a second.

3. A boy is running on a horizontal plane in a straight line towards the base of a tower 50 metres in height. How fast is he approaching the top when he is 500 metres from the foot, and is running at the rate of 200 metres a minute?

Ans. 199 metres a minute.

4. A light is 4 metres above and directly over a straight horizontal sidewalk, on which a man $1\frac{2}{3}$ metres in height is walking away from the light at the rate of 50 metres a minute. How fast is his shadow changing in length?



Let AE be the sidewalk, B the position of the light, and CD one position of the man.

Let $y =$ the number of metres in CE,
and $x =$ the number in AC; then
$y + x : y = 4 : 5/3$, etc.

5. In problem 4 show that the farthest point of the man's shadow is moving at the rate of $85\frac{5}{7}$ metres a minute.

6. The altitude of a variable cylinder is constantly equal to the diameter of the base. When the altitude equals 6 metres it is increasing at the rate of 2 metres an hour; how fast is its volume increasing? How fast its entire surface?

Ans. 54π kilolitres an hour; 36π centiares an hour.

7. Find the length of a side of an equilateral triangle when its area is increasing in sq. in. 30 times as fast as a side is in lin. in.

Let $x =$ the number of lin. in. in a side of the triangle,
and $y =$ the number of sq. in. in its area;
then $y = \frac{1}{4}\sqrt{3}\,x^2$; $\therefore dy = \frac{1}{2}\sqrt{3}\,x dx$.

Hence, y changes $\frac{1}{2}\sqrt{3}\,x$ times as fast as x; therefore, when y changes 30 times as fast as x, $\frac{1}{2}\sqrt{3}\,x = 30$, or $x = 20\sqrt{3}$.

8. On the parabola $y^2 = 8x$, find the point at which y changes at the rate of 2 to 1 of x.

Ans. $(1/2,\ 2)$.

9. On the ellipse $x^2/a^2 + y^2/b^2 = 1$, find the points at which y changes at the rate of c to 1 of x.

Ans. $(\mp ca^2/\sqrt{b^2 + a^2c^2},\ \pm b^2/\sqrt{b^2 + a^2c^2})$.

10. One ship was sailing south at the rate of 6 miles an hour; another east at the rate of 8 miles an hour. At 4 P.M. the second crosses the track of the first at a point where the first was two hours before. When was the distance between the ships not changing? How was it changing at 3 P.M.? at 5 P.M.?

Let $t =$ the number of hours in the time reckoned from 4 P.M., time after 4 P.M. being +, and time before −. Then $8t$ miles and $(6t + 12)$ miles will be respectively the distances of the two ships from the point of intersection of their paths, distances south and east being +, and distances west and north being −.

Let $y =$ the number of miles between the ships; then

$$y^2 = (8t)^2 + (6t + 12)^2.$$

$$\therefore \frac{dy}{dt} = \frac{100t + 72}{[64t^2 + (6t + 12)^2]^{1/2}}.$$

When $dy/dt = 0$, $100t + 72 = 0$, or $t = -0.72$.

Hence, the distance between the ships was not changing at 43.2 minutes before 4 P.M., or at 16.8 minutes after 3 P.M.

Ans. Diminishing at the rate of 2.8 miles an hour; increasing at the rate of 8.73 miles an hour.

11. Two straight lines of railway intersect at an angle of 120°; on one line a train is approaching the intersection at the rate of 30 miles an hour, and on the other a train is receding from it at the rate of 40 miles an hour. When the first is 10 miles from the intersection, the second is 20 miles from it; find the rate of change of the distance between the trains at that instant.

> Let $z =$ the number of miles between the trains, and x and y respectively the numbers of miles between them and the intersection of the tracks; then the rates of x and y respectively will be -30 and 40 an hour, and
>
> $$z^2 = x^2 + y^2 + xy.$$
>
> $$\therefore \frac{dz}{dt} = \frac{1}{2z}\left\{(2x+y)\frac{dx}{dt} + (2y+x)\frac{dy}{dt}\right\} = \frac{40}{7}\sqrt{7}.$$
>
> Hence, at the instant considered the trains are separating at the rate of $\frac{40}{7}\sqrt{7}$ miles an hour.

31. Tangent. If a secant RS be moved or revolved so that two of its points of intersection with a curve coincide at some point P, the line RS in this position is the tangent to the curve at P.

32. Slope. *The tangent to a curve at any point P has the direction of the curve at that point.

Let ϕ denote the angle XBP (§ 33, fig.); then, when P moves along the curve, ϕ varies and gives the direction of the curve at P relative to the x-axis.

Tan ϕ is called the **slope** of the curve at P.

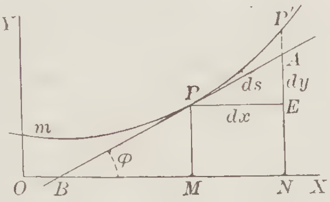
33. Geometric meaning of dy / dx. Let mPP' be the locus of $y = f(x)$. Let s denote the length of the path traced by the point (x, y).

---

[*] This statement, if not sufficiently evident, may be proved as follows: When the direction of the arc PaP' (§ 60, fig.) changes continuously (this arc can always be made so small that its direction will change continuously), the secant PP' has the direction of the arc PaP' at some point, as a. When P' is made to coincide with P, a also will coincide with P; hence, the tangent at P has the direction of the curve at P.

Starting at m, suppose (x, y) to move along the curve to P and thence along the tangent PA. Then, when $x = OM$, the change of x and that of y will both become uniform with respect to s.

Hence, PA, PE, and EA will represent ds, dx, and dy, respectively, when $x = OM$.



Hence,
$$dy/dx = \tan\phi. \tag{1}$$

That is, dy/dx, *or* $f'(x)$, *equals the slope of* $y = f(x)$ *at* (x, y).

Since the relative rates of y and x determine the direction of motion of the point (x, y), equation (1) follows directly from § 12.

Cor. 1. From the right angled triangle PEA we have

$$ds^2 = dx^2 + dy^2, \tag{2}$$

$$dy/ds = \sin\phi, \qquad dx/ds = \cos\phi. \tag{3}$$

Cor. 2. If PA represents the velocity at P of the point (x, y) in the direction of its path, PE and EA will represent the component velocities at P of the point (x, y) in the direction of the axes.

The figure given above illustrates the difference between an increment and a differential.

For example, if $\Delta x = PE = dx$, $\Delta y = EP' = dy + AP'$.

If the curve were concave downward at P, Δy would be less than dy. Δy and dy are equal when and only when the locus is a straight line.

34. Rectangular equation of a tangent. Let $y = f(x)$ be the rectangular equation of any plane curve, and let dy'/dx' denote the value of dy/dx for the point (x', y'); then the equation of the tangent to the curve $y = f(x)$ at the point (x', y') is

$$y - y' = \frac{dy'}{dx'}(x - x'). \tag{a}$$

For line (a) evidently passes through (x', y'), and by § 33 it has the slope of the curve at that point.

Ex. Find the equation of the tangent to the parabola $y^2 = 4px$.

Here $dy/dx = 2p/y$; $\therefore dy'/dx' = 2p/y'$.

Substituting in (a), we obtain as the required equation

$$y - y' = \frac{2p}{y'}(x - x'). \tag{1}$$

Since $y'^2 = 4px'$, equation (1) becomes by reduction

$$yy' = 2p(x + x'). \tag{2}$$

Cor. Intercept of tangent on x-axis $= x' - y'dx'/dy'$. (1)

Intercept of tangent on y-axis $= y' - x'dy'/dx'$. (2)

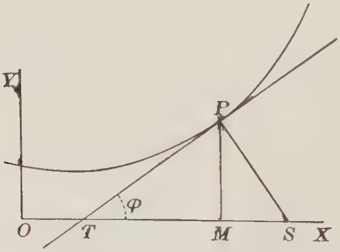
35. Rectangular equation of a normal. The normal at the point (x', y') passes through (x', y') and is perpendicular to the tangent at that point; hence, its equation is

$$y - y' = -\frac{dx'}{dy'}(x - x'). \tag{b}$$

For example, the equation of the normal to the parabola $y^2 = 4px$ at the point (x', y') is

$$y - y' = -(y'/2p)(x - x').$$

36. Subtangent, subnormal, tangent, normal. Let PT be the tangent at the point $P(x', y')$, and PS the normal.



Draw the ordinate PM; then TM is called the *subtangent*, and MS the *subnormal*. Hence,

$$\text{Subt.} = TM = MP\cot\phi \qquad = y'dx'/dy'. \tag{1}$$

$$\text{Subn.} = MS = MP\tan\phi \qquad = y'dy'/dx'. \tag{2}$$

$$\text{Tan.} = TP = \sqrt{\overline{MP}^2 + \overline{TM}^2} = y'\sqrt{1 + (dx'/dy')^2}. \tag{3}$$

$$\text{Norm.} = SP = \sqrt{\overline{MP}^2 + \overline{MS}^2} = y'\sqrt{1 + (dy'/dx')^2}. \tag{4}$$

If the subtangent is reckoned from the point T, and the subnormal from M, each will be positive or negative according as it extends to the right or to the left.

NOTE. The problem of tangents was foremost among the problems which led to the invention of the Differential Calculus.

EXAMPLES.

The equations of the tangent and the normal to

1. The circle, $x^2 + y^2 = r^2$, are $yy' + xx' = r^2$, and $x'y = y'x$.

2. The ellipse, $x^2/a^2 + y^2/b^2 = 1$, are $xx'/a^2 + yy'/b^2 = 1$, and
$$y - y' = (a^2y'/b^2x')(x - x').$$

3. The hyperbola, $x^2/a^2 - y^2/b^2 = 1$, are $xx'/a^2 - yy'/b^2 = 1$, and
$$y - y' = -(a^2y'/b^2x')(x - x').$$

4. The cissoid, $y^2 = \dfrac{x^3}{2a - x}$, are $y - y' = \pm\dfrac{\sqrt{x'}\,(3a - x')}{(2a - x')^{3/2}}(x - x')$,
$$\text{and } y - y' = \mp\frac{(2a - x')^{3/2}}{\sqrt{x'}\,(3a - x')}(x - x'). \quad \text{§ 155, fig. 3.}$$

5. The hyperbola, $xy = m$, are $x'y + y'x = 2m$, and
$$y'y - x'x = y'^2 - x'^2.$$

6. The circle, $x^2 + y^2 = 2rx$, are $y - y' = (x - x')(r - x')/y'$, and
$$y - y' = (x - x')y'/(x' - r).$$

7. Find the equations of the tangent and the normal to the curve $y^2 = 2x^2 - x^3$, (1) at the point whose abscissa is 1, (2) at the point whose abscissa is -2.

8. Find the slope of the curve $y^2 = x^3 + 2x^4$ at $x = 2$.

9. Find the slope of the curve $y = x^3 - y^2 + 1$ at $x = 2$; $x = 1$; $x = 0$; $x = -1$.

10. At what point on $y^2 = 2x^3$ is the slope 3? At what point is the curve parallel to the x-axis? *Ans.* (2, 4); (0, 0).

11. At what angle does $y^2 = 8x$ intersect $4x^2 + 2y^2 = 48$?

> The points of intersection are (2, 4) and (2, − 4); the slopes of the curves are $4/y$ and $-2x/y$ respectively, which for the point (2, 4) become 1 and − 1. Hence, the curves intersect at right angles at (2, 4).

12. At what angles does the line $3y - 2x - 8 = 0$ cut the parabola $y^2 = 8x$? *Ans.* $\tan^{-1} 0.2$; $\tan^{-1} 0.125$.

13. The cissoid $y^2 = x^3/(2a - x)$ cuts its circle $x^2 + y^2 = 2ax$ at $\tan^{-1} 2$.

14. Find the subtangents and the subnormals of the conic sections and the cissoid.

Ans. Parabola: subt. $= 2x'$; subn. $= 2p$.
Ellipse: subt. $= (x'^2 - a^2)/x'$; subn. $= -b^2x'/a^2$.
Hyperbola: subt. $= (x'^2 - a^2)/x'$; subn. $= b^2x'/a^2$.
Cissoid: subt. $= \dfrac{x'(2a - x')}{3a - x'}$; subn. $= \dfrac{x'^2(3a - x')}{(2a - x')^2}$.

Find the subtangent and the subnormal of

15. The hyperbola $xy = m$.

16. The semi-cubical parabola $ay^2 = x^3$. § 155, fig. 2.

17. Find the slope of the logarithmic curve $x = \log_a y$. The slope varies as what? In the curve $x = \log y$ the slope equals what?

18. Find the equations of the tangent and the normal to $x = \log_a y$. Show that the subt. $= m$, and find the subn.

19. Find the normal, subnormal, tangent, and subtangent of the catenary $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$. §197, fig.

$$Ans.\quad \frac{y^2}{a};\ \frac{a}{4}(e^{2x/a} - e^{-2x/a});\ \frac{y^2}{\sqrt{y^2 - a^2}};\ \frac{ay}{\sqrt{y^2 - a^2}}.$$

20. The path of a point is an arc of the parabola $y^2 = 4px$, and its velocity is v; find its velocity in the direction of each axis.

Let s denote the length of the path measured from any point upon it; then

$$ds / dt = v.$$

From $y^2 = 4px$, $$\frac{dy}{dt} = \frac{2p}{y} \frac{dx}{dt}.$$

Substituting these values in

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \qquad \text{§ 33, Cor.}$$

we obtain

$$\frac{dx}{dt} = \frac{yv}{\sqrt{y^2 + 4p^2}}, \text{ and } \frac{dy}{dt} = \frac{2pv}{\sqrt{y^2 + 4p^2}}.$$

21. Find the velocities required in example 20, when the path is,

(i) an arc of the circle $x^2 + y^2 = r^2$,

(ii) an arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

(iii) an arc of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

22. A comet's orbit is a parabola, and its velocity is v; find its rate of approach to the sun, which is at the focus of its orbit.

Let ρ denote the distance from the focus to any point on $y^2 = 4px$; then

$$\rho = x + p.$$

$$\therefore d\rho / dt = dx / dt. \qquad (1)$$

Hence, the comet approaches or recedes from the sun just as fast as it moves parallel to the axis of its orbit.

$$\therefore \frac{d\rho}{dt} = \frac{dx}{dt} = \frac{y}{\sqrt{y^2 + 4p^2}} v. \qquad \text{Example 20}$$

At the vertex $y = 0$; hence, at the vertex $d\rho / dt$ is zero.

When $y = 2p$, $d\rho / dt = (1/2)\sqrt{2}\, v$.

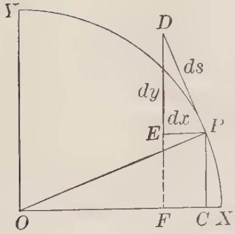
When $y = 3p$, $d\rho / dt = (3/13)\sqrt{13}\, v$.

23. The curves $y = f(x)$ and $y = F(x)$ have the point $[a, f(a)]$ in common; show that these curves intersect at this point at an angle whose tangent is $\frac{f'(a) - F'(a)}{1 + f'(a) \cdot F'(a)}$.

Trigonometric Functions.

37. A **radian** is an angle which, when placed at the centre of a circle, intercepts an arc a radius in length. It equals $180°/\pi$, or $57°.3$ nearly. Let u denote the number of radians in any angle at the centre of a circle, r the number of units in the radius, and s the number in the intercepted arc; then

$$u = s/r; \quad \text{or if} \quad r = 1, \quad u = s.$$



38. $\mathbf{d^* \sin u \equiv \cos u\, du.}$ [17]

$\mathbf{d \cos u \equiv - \sin u\, du.}$ [18]

Let the point $P\,(x, y)$ move along the arc XPY of a unit circle. Denote the number of linear units in the arc XP by s, and the number of *radians* in the angle XOP by u. We shall then have

$$u = s, \qquad y = \sin u, \qquad x = \cos u.$$

$$\therefore du = ds, \qquad dy = d \sin u, \qquad dx = d \cos u. \tag{1}$$

Angle EDP equals u, and dx is negative; hence, from the triangle EDP by § 33 we obtain

$$dy = \cos u\, ds, \qquad dx = - \sin u\, ds. \tag{2}$$

Substituting for dy, ds, and dx in (2) their values as given in (1), we obtain [17] and [18].

39. $\mathbf{d \tan u \equiv \sec^2 u\, du.}$ [19]

For $\tan u = \sin u / \cos u.$

$$\therefore d \tan u = \frac{\cos u\, d \sin u - \sin u\, d \cos u}{\cos^2 u}$$

$$= \frac{(\cos^2 u + \sin^2 u)\, du}{\cos^2 u} = \sec^2 u\, du.$$

---

* When not needed to avoid ambiguity the parentheses after the sign *d* are often omitted.

40. $\mathbf{d \cot u \equiv -\csc^2 u\, du.}$ [20]

For $\cot u = \tan(\pi/2 - u).$

$$\therefore d\cot u = \sec^2(\pi/2 - u)\, d(\pi/2 - u)$$
$$= -\csc^2 u\, du.$$

41. $\mathbf{d \sec u \equiv \sec u \tan u\, du.}$ [21]

For $\sec u = 1/\cos u.$

$$\therefore d\sec u = \frac{\sin u\, du}{\cos^2 u} = \sec u \tan u\, du.$$

42. $\mathbf{d \csc u \equiv -\csc u \cot u\, du.}$ [22]

For $\csc u = \sec(\pi/2 - u).$

$$\therefore d\csc u = \sec(\pi/2 - u)\tan(\pi/2 - u)\, d(\pi/2 - u)$$
$$= -\csc u \cot u\, du.$$

43. $\mathbf{d\, \mathrm{vers}\, u} \equiv d(1 - \cos u) \equiv \mathbf{\sin u\, du.}$ [23]

44. $\mathbf{d\, \mathrm{covers}\, u} \equiv d(1 - \sin u) \equiv \mathbf{-\cos u\, du.}$ [24]

EXAMPLES.

1. $y = \sin ax.$ $dy = a\cos ax \cdot dx.$

2. $y = \cos(x/a).$ $dy = -a^{-1}\sin(x/a)\, dx.$

3. $y = \cos x^3.$ $dy = -3x^2 \sin x^3 dx.$

4. $f(\theta) = \tan^m \theta.$ $f'(\theta) = m\tan^{m-1}\theta \sec^2\theta.$

5. $f(\theta) = \tan 3\theta + \sec 3\theta.$ $f'(\theta) = 3\sec^2 3\theta + 3\sec 3\theta \tan 3\theta.$

6. $f(x) = \sin(\log ax).$ $f'(x) = x^{-1}\cos(\log ax).$

$f(x)$ changes at the rate of $f'(x)$ to 1 of x (§ 12).

7. $f(x) = \log(\sin ax).$ $f'(x) = a\cot ax.$

8. $f(\theta) = \log(\tan a\theta).$ $f'(\theta) = \dfrac{2a}{2\sin a\theta \cos a\theta} = \dfrac{2a}{\sin 2a\theta}.$

9. $f(\theta) = \log(\cot a\theta).$ $f'(\theta) = -2a/\sin 2a\theta.$

10. $f(x) = \dfrac{1 - \tan ax}{\sec ax} = \cos ax - \sin ax.$

11. $f(x) = x^n e^{\sin x}.$ $\qquad f'(x) = x^{n-1} e^{\sin x}(n + x \cos x).$

12. $f(x) = \sin nx \cdot \sin^n x.$ $\qquad f'(x) = n \sin^{n-1} x \cdot \sin(nx + x).$

13. $f(x) = e^x \log \sin x.$ $\qquad f'(x) = e^x(\cot x + \log \sin x).$

14. $f(x) = \tan(\log x).$ $\qquad$ 15. $f(x) = \log \sec x.$

16. $f(x) = \cos x / 2 \sin^2 x.$

17. $f(x) = 4 \sin^m ax.$ $\qquad f'(x) = 4\, am \sin^{m-1} ax \cos ax.$

18. $f(x) = x^{\sin x}.$ $\qquad f'(x) = x^{\sin x}(\sin x / x + \log x \cos x).$

19. $f(\theta) = (\sin \theta)^{\tan \theta}.$ $\qquad f'(\theta) = (\sin \theta)^{\tan \theta}(1 + \sec^2 \theta \log \sin \theta).$

20. $f(x) = \tan a^{1/x}.$ $\qquad f'(x) = -a^{1/x} \sec^2 a^{1/x} \log a / x^2.$

21. $f(\theta) = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta.$ $\qquad f'(\theta) = \tan^4 \theta.$

22. $f(x) = e^{(a+x)^2} \sin x.$ $\qquad f'(x) = e^{(a+x)^2}[2(a + x) \sin x + \cos x].$

23. $f(x) = e^{-a^2x^2} \cos rx.$ $\qquad f'(x) = -e^{-a^2x^2}(2a^2x \cos rx + r \sin rx).$

24. $f(x) = \tan \sqrt{1 - x}.$ $\qquad f'(x) = \dfrac{-(\sec \sqrt{1 - x})^2}{2\sqrt{1 - x}}.$

25. $f(\theta) = \log \sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}}.$ $\qquad f'(\theta) = \csc \theta.$

By differentiation derive each of the following pairs of identities from the other:

26. $\sin 2\theta \equiv 2 \sin \theta \cos \theta, \ \cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta.$

27. $\sin 2\theta \equiv \dfrac{2 \tan \theta}{1 + \tan^2 \theta}, \ \cos 2\theta \equiv \dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}.$

28. $\sin 3\theta \equiv 3 \sin \theta - 4 \sin^3 \theta,$
$\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta.$

29. $\sin(m + n)\theta \equiv \sin m\theta \cos n\theta + \cos m\theta \sin n\theta,$
$\cos(m + n)\theta \equiv \cos m\theta \cos n\theta - \sin m\theta \sin n\theta.$

30. $df(a + x^2) = f'(a + x^2)\, 2\, xdx.$

31. $df(ax^2) = f'(ax^2)\, 2\, axdx.$

32. $\dfrac{d}{dx} f\left(\dfrac{x^2}{a}\right) = f'\left(\dfrac{x^2}{a}\right) \dfrac{2x}{a}.$

33. $df(xy) = f'(xy)(xdy + ydx).$

Inverse-Trigonometric Functions.

45. $\mathbf{d \sin^{-1} u \equiv du / \sqrt{1 - u^2}}$.* [25]

That is, *the differential of an angle in terms of its sine is the differential of the sine divided by the square root of one minus the square of the sine.*

Let $y = \sin^{-1} u$; then $\sin y = u$.

$$\therefore dy = \frac{du}{\cos y} = \frac{du}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{du}{\sqrt{1 - u^2}}.$$

46. $\mathbf{d \cos^{-1} u} \equiv d\left(\frac{\pi}{2} - \sin^{-1} u\right) \equiv -\frac{\mathbf{du}}{\sqrt{\mathbf{1 - u^2}}}.$ [26]

47. $\mathbf{d \tan^{-1} u \equiv \frac{du}{1 + u^2}}.$ [27]

Let $y = \tan^{-1} u$; then $\tan y = u$.

$$\therefore dy = \frac{du}{\sec^2 y} = \frac{du}{1 + \tan^2 y} = \frac{du}{1 + u^2}.$$

48. $\mathbf{d \cot^{-1} u} \equiv d\left(\frac{\pi}{2} - \tan^{-1} u\right) \equiv -\frac{\mathbf{du}}{\mathbf{1 + u^2}}.$ [28]

49. $\mathbf{d \sec^{-1} u \equiv \frac{du}{u\sqrt{u^2 - 1}}}.$ [29]

Let $y = \sec^{-1} u$; then $\sec y = u$.

$$\therefore dy = \frac{du}{\sec y \tan y} = \frac{du}{u\sqrt{u^2 - 1}}$$

---

* To avoid the ambiguity of the double sign ±, we shall in these formulas limit $\sin^{-1} u$, $\cos^{-1} u$, etc., to values between 0 and $\pi/2$.

50. $$\mathbf{d\,csc^{-1}u} \equiv d\left(\frac{\pi}{2} - \sec^{-1}u\right) \equiv -\frac{\mathbf{du}}{\mathbf{u\sqrt{u^2-1}}}. \qquad [30]$$

51. $$\mathbf{d\,vers^{-1}u} \equiv \frac{\mathbf{du}}{\mathbf{\sqrt{2\,u-u^2}}}. \qquad [31]$$

Let $y = \text{vers}^{-1}u$; then $\text{vers}\, y = u$.

$$\therefore dy = \frac{du}{\sin y} = \frac{du}{\sqrt{1-\cos^2 y}}$$

$$= \frac{du}{\sqrt{1-(1-\text{vers}\, y)^2}}$$

$$= \frac{du}{\sqrt{1-(1-u)^2}} = \frac{du}{\sqrt{2\,u-u^2}}.$$

52. $$\mathbf{d\,covers^{-1}u} \equiv d\left(\frac{\pi}{2} - \text{vers}^{-1}u\right) \equiv -\frac{\mathbf{du}}{\mathbf{\sqrt{2\,u-u^2}}}. \qquad [32]$$

EXAMPLES.

1. $d\sin^{-1}\frac{x}{a} = \frac{d(x/a)}{\sqrt{1-(x/a)^2}} = \frac{dx}{\sqrt{a^2-x^2}}.$

2. $d\cos^{-1}\frac{x}{a} = -\frac{dx}{\sqrt{a^2-x^2}};$ $\qquad d\tan^{-1}\frac{x}{a} = \frac{adx}{a^2+x^2}.$

$d\cot^{-1}\frac{x}{a} = -\frac{adx}{a^2+x^2};$ $\qquad d\sec^{-1}\frac{x}{a} = \frac{adx}{x\sqrt{x^2-a^2}};$

$d\csc^{-1}\frac{x}{a} = -\frac{adx}{x\sqrt{x^2-a^2}};$ $\qquad d\,\text{vers}^{-1}\frac{x}{a} = \frac{dx}{\sqrt{2\,ax-x^2}}.$

3. $y = x\sin^{-1}mx.$ $\qquad \frac{dy}{dx} = \sin^{-1}mx + \frac{mx}{\sqrt{1-m^2x^2}}.$

4. $y = \tan x \tan^{-1}x.$ $\qquad \frac{dy}{dx} = \sec^2 x \tan^{-1}x + \frac{\tan x}{1+x^2}.$

5. $y = \tan^{-1}\frac{2\,x}{1+x^2}.$ $\qquad \frac{dy}{dx} = \frac{2(1-x^2)}{1+6\,x^2+x^4}.$

6. $y = \sin^{-1}\frac{x+1}{\sqrt{2}}.$ $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-2x-x^2}}.$

7. $y = \sec^{-1}\frac{1}{2x^2-1}.$ $\qquad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}.$

8. $y = \cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}.$ $\qquad \frac{dy}{dx} = -\frac{2nx^{n-1}}{x^{2n}+1}.$

9. $y = \tan^{-1}(n\tan x).$ $\qquad \frac{dy}{dx} = \frac{n}{\cos^2 x + n^2\sin^2 x}.$

10. $y = x^{\sin^{-1}x}.$ $\qquad \frac{dy}{dx} = x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right).$

11. $y = \tan^{-1}\frac{x+a}{1-ax}.$ $\qquad \frac{dy}{dx} = \frac{1}{1+x^2}.$

12. $y = \sin^{-1}\sqrt{\sin x}.$ $\qquad \frac{dy}{dx} = \frac{\sqrt{1+\csc x}}{2}.$

13. $y = \cos^{-1}\frac{e^x-e^{-x}}{e^x+e^{-x}}.$ $\qquad \frac{dy}{dx} = \frac{-2}{e^x+e^{-x}}.$

14. $y = \tan^{-1}\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}.$ $\qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x)}.$

15. $y = \sec^{-1}\sqrt{\frac{2}{1+x}}.$ $\qquad \frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}.$

16. $y = \cot^{-1}\frac{1+\sqrt{1+x^2}}{x}.$ $\qquad \frac{dy}{dx} = \frac{1}{2(1+x^2)}.$

17. $y = \cot^{-1}\sqrt{\frac{1-x}{2+x}}.$ $\qquad \frac{dy}{dx} = \frac{1}{2\sqrt{2-x-x^2}}.$

18. $y = \operatorname{vers}^{-1}\frac{2x^2}{1+x^2}.$ $\qquad \frac{dy}{dx} = \frac{2}{1+x^2}.$

19. $y = \tan^{-1}\frac{3x-x^3}{1-3x^2}.$ $\qquad \frac{dy}{dx} = \frac{3}{1+x^2}.$

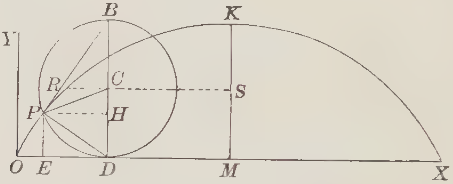
20. $y = \csc^{-1}\frac{1}{2x^2-1}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}.$

21. $y = \sin^{-1}\frac{1-x^2}{1+x^2}.$ $\qquad \frac{dy}{dx} = \frac{-2}{1+x^2}.$

22. $y = \cos^{-1}\dfrac{3 + 5\cos x}{5 + 3\cos x}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{5 + 3\cos x}$.

23. $y = \tan^{-1}\dfrac{x}{\sqrt{1 - x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

24. A wheel whose radius is r rolls along a horizontal line with a velocity v; find the velocity of any point P in its rim, also the velocity of P horizontally and vertically.



The path of P is a cycloid whose equations are

$$\left.\begin{aligned} x &= r(\theta - \sin\theta), \\ y &= r(1 - \cos\theta) = r \operatorname{vers}\theta, \end{aligned}\right\} \tag{1}$$

where θ denotes the variable angle DCP, and r the radius CD.

Since the centre of the wheel is vertically over D,

$$\begin{aligned} v &= \text{the time-rate of } OD \\ &= d(r\theta)/dt = r \cdot d\theta/dt. \end{aligned}$$

$$\therefore\ d\theta/dt = v/r. \tag{2}$$

Differentiating equations (1), by (2) we obtain

$$dx/dt = v \cdot \operatorname{vers}\theta = \text{the velocity horizontally}, \tag{3}$$

and $$dy/dt = v \cdot \sin\theta \quad = \text{the velocity vertically}. \tag{4}$$

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= v\sqrt{2(1 - \cos\theta)} = v\sqrt{2y/r} \\ &= \text{the velocity of } P \text{ along its path.} \end{aligned} \tag{5}$$

$$\text{At } O,\ \theta = 0, \text{ and } \frac{dx}{dt} = \frac{dy}{dt} = \frac{ds}{dt} = 0.$$

$$\text{At } R,\ \theta = \frac{\pi}{2}, \qquad \frac{dx}{dt} = \frac{dy}{dt} = v, \qquad \frac{ds}{dt} = v\sqrt{2}.$$

$$\text{At } K,\ \theta = \pi, \qquad \frac{dx}{dt} = \frac{ds}{dt} = 2v, \quad \frac{dy}{dt} = 0.$$

From (5) we obtain

$$ds/dt : v = \sqrt{2r \cdot y} : r.$$

Hence, the velocity of P is to that of C as the chord DP is to the radius DC; that is, P and C are momentarily moving about D with equal angular velocities.

25. Find the subnormal and the normal of the cycloid.

$$\text{Subn.} = r \sin\theta = PH = ED.$$

Thus the normal at P passes through the foot of the perpendicular to OM from C. Hence, to draw a tangent and normal at P, locate C, draw the perpendicular DCB equal to $2r$, and join P with B and D; then PB and PD will be respectively the tangent and the normal at P.

$$\text{Normal} = DP = \sqrt{DB \cdot DH} = \sqrt{2r \cdot y}.$$

26. Eliminating θ in equations (1) of example 24, find the equation of the cycloid in the form

$$x = r \operatorname{vers}^{-1}(y/r) \mp \sqrt{2ry - y^2}.$$

27. The equation of the tangent to the cycloid is

$$y - y' = \sqrt{(2r - y')/y'}\ (x - x').$$

28. A vertical wheel whose circumference is 20 ft. makes 5 revolutions a second about a fixed axis. How fast is a point in its circumference moving horizontally when it is 30° from either extremity of the horizontal diameter? *Ans.* 50 ft. a second.

29. What is the slope of the curve $y = \sin x$? Its inclination lies between what values? What is its inclination at $x = 0$? What at $x = \pi/2$?

The slope $= \cos x$; hence, at any point, it must be something between -1 and $+1$ inclusive. Hence, the inclination of the curve at any point is something between 0 and $\pi/4$, or something between $3\pi/4$ and π inclusive.

30. Find the equation of the tangent to the curve $y = \sin x$; $y = \tan x$; $y = \sec x$.

MISCELLANEOUS EXAMPLES.

1. $y = \log \tan^{-1} x.$

2. $y = (x + \sqrt{1 - x^2})^n.$

3. $y = \sqrt{\dfrac{1 - x^2}{(1 + x^2)^3}}.$

4. $y = \dfrac{(\sin nx)^m}{(\cos mx)^n}.$

5. $y = e^{x^x} \tan^{-1} x.$

6. $y = \sin^{-1} \dfrac{3 + 2x}{\sqrt{13}}.$

7. $f(x) = e^{(a+x)^2} \sin x.$

8. $y = \dfrac{x \log x}{1 - x} + \log (1 - x).$

9. $y = \sec^{-1} \dfrac{a}{\sqrt{a^2 - x^2}}.$

10. $y = e^{ax} \sin^m rx.$

11. $y = \log \{\log (a + bx^n)\}.$

12. $y = \dfrac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}.$

In example 12 rationalize the denominator before differentiating.

13. $f(x) = (a^2 + x^2) \tan^{-1} \dfrac{x}{a}.$

14. $y = \sqrt{1 - x^2} \sin^{-1} x - x.$

15. $y = \tan^{-1} \dfrac{x}{a} + \log \sqrt{\dfrac{x - a}{x + a}}.$

16. $y = \dfrac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} - x}.$

17. $y = \dfrac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}.$

$$\frac{dy}{dx} = \frac{2x}{a^2 - b^2}\left(2 + \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} + \sqrt{\frac{x^2 + b^2}{x^2 + a^2}}\right).$$

18. $y = \log (\sqrt{1 + x^2} + \sqrt{1 - x^2}).$
$$\frac{dy}{dx} = \frac{1}{x}\left(1 - \frac{1}{\sqrt{1 - x^4}}\right).$$

19. $f(x) = (x - 3) e^{2x} + 4xe^x + x + 3.$

20. $y = \log (2x - 1 + 2\sqrt{x^2 - x - 1}).$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - x - 1}}.$$

21. $y = \log \left(\dfrac{1 + x}{1 - x}\right)^{1/4} - \dfrac{\tan^{-1} x}{2}.$
$$\frac{dy}{dx} = \frac{x^2}{1 - x^4}.$$

22. $x = e^{\frac{x - y}{y}}.$
$$\frac{dy}{dx} = \frac{x - y}{x(\log x + 1)}.$$

Here $\log x = \dfrac{x - y}{y}.$

23. $y = \cfrac{x^2}{1 + \cfrac{x^2}{1 + \cfrac{x^2}{1 + \text{etc. to infinity.}}}}$

Here $y = \dfrac{x^2}{1+y}$.

24. $y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}\dfrac{x}{a}$. $\quad \dfrac{dy}{dx} = \dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}}$.

25. $y = \log \dfrac{\sqrt{1-x^2} + x\sqrt{2}}{\sqrt{1-x^2}}$. $\quad \dfrac{dy}{dx} = \dfrac{\sqrt{2}}{(\sqrt{1-x^2} + x\sqrt{2})(1-x^2)}$.

26. $y = \log\sqrt{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$. $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$.

Here $y = \log(\sqrt{1+x^2} + x)$.

27. $y = \log\sqrt{\dfrac{2x^2 - 2x + 1}{2x^2 + 2x + 1}} + \tan^{-1}\dfrac{2x}{1-2x^2}$. $\quad \dfrac{dy}{dx} = \dfrac{8x^2}{4x^4 + 1}$.

28. $y = \cot^{-1}\dfrac{a}{x} + \log\sqrt{\dfrac{x-a}{x+a}}$. $\quad \dfrac{dy}{dx} = \dfrac{2ax^2}{x^4 - a^4}$.

29. $y = \sin^{-1}\dfrac{x\tan\theta}{\sqrt{a^2 - x^2}}$. $\quad \dfrac{dy}{dx} = \dfrac{a^2\tan\theta}{a^2 - x^2}\,\dfrac{1}{\sqrt{a^2 - x^2\sec^2\theta}}$.

30. $y = \log\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{3/4}\right\}$. $\quad \dfrac{dy}{dx} = \dfrac{x^2-1}{x^2-4}$.

CHAPTER III.

PROBLEM OF RATES SOLVED BY LIMITS.

53. Limit. When according to its law of change a variable approaches indefinitely near and continually nearer a constant, but can never reach it, the constant is called the *limit* of the variable.

It is assumed that the reader is familiar with the elementary theorems of Limits; but for convenience of reference we state them below:

1. If two variables are equal, their limits are equal.
2. The limit of the sum, or product, of a constant and a variable is the sum, or product, of the constant and the limit of the variable.
3. The limit of the variable sum, or product, of two or more variables is the sum, or product, of their limits.
4. The limit of the variable quotient of two variables is the quotient of their limits, **except when the limit of the divisor is zero.**

54. Notation. The sign $\doteq$ denotes "approaches as a limit." Thus, $\Delta x \doteq 0$ is read "Δx approaches zero as its limit."

The limit of a variable, as z, is often written lt (z).

$\underset{\Delta x \doteq 0}{\text{Limit}}\left[\frac{\Delta y}{\Delta x}\right]$, or lt $\frac{\Delta y}{\Delta x}$, denotes lt $\left(\frac{\Delta y}{\Delta x}\right)$ when $\Delta x \doteq 0$.

55. $$\underset{\Delta x \doteq 0}{\textbf{Limit}}\left[\frac{\Delta y}{\Delta x}\right] \equiv \frac{dy}{dx}. \qquad [33]$$

Let v', x', and y' denote any corresponding values of v, x, and y, from which Δv, Δx, and Δy are estimated. Let the rate of v be the unit of rates; then evidently

$$\frac{\Delta y}{\Delta v} = \left\{\begin{matrix}\text{the } v\text{-rate of } y \text{ at some value of } y \\ \text{between } y' \text{ and } y' + \Delta y\end{matrix}\right\}. \qquad (1)$$

$$\therefore \text{lt}\,(\Delta y/\Delta v) = \text{the } v\text{-rate of } y \text{ at the value } y'. \qquad (2)$$

Also $$\text{lt}\,(\Delta x/\Delta v) = \text{the } v\text{-rate of } x \text{ at the value } x'. \qquad (3)$$

Dividing (2) by (3), we obtain, in general,

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right] = \frac{\text{the } v\text{-rate of } y}{\text{the } v\text{-rate of } x}. \qquad (4)$$

Comparing (4) with (1) of § 12, we obtain [33].

To illustrate (1) and (2), let s denote the number of feet a falling body descends in t seconds. Let s' and t' denote any corresponding values of s and t, from which Δs and Δt are reckoned; then, evidently,

$$\frac{\Delta s}{\Delta t} = \left\{\begin{array}{c}\text{the time-rate of } s \text{ at some value of } s \\ \text{between } s' \text{ and } s' + \Delta s\end{array}\right\}.$$

$$\therefore \text{lt}\,(\Delta s/\Delta t) = \text{the time-rate of } s \text{ at the value } s'.$$

EXAMPLES.

1. Prove $\Delta \mathbf{f}(\mathbf{u}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{u} + \Delta\mathbf{u}, \mathbf{y} + \Delta\mathbf{y}) - \mathbf{f}(\mathbf{u}, \mathbf{y})$, (b)

where u and y are functions of x.

When $x = x'$, then $u = u'$, $y = y'$, and $f(u, y) = f(u', y')$.

When $x = x' + \Delta x$, then $u = u' + \Delta u$, $y = y' + \Delta y$,

and $$f(u, y) = f(u' + \Delta u, y' + \Delta y).$$

Hence, while x changes from x' to $x' + \Delta x$, $f(u, y)$ changes from $f(u', y')$ to $f(u' + \Delta u, y' + \Delta y)$; that is, if Δx is reckoned from the value x',

$$\Delta f(u, y) = f(u' + \Delta u, y' + \Delta y) - f(u', y'). \qquad (1)$$

Since x', u', and y' are any corresponding values of x, u, and y, from (1) we have (b).

2. Prove $d(uy) = y\,du + u\,dy$. [7]

By formula (1) in example 1, we have

$$\begin{aligned}\Delta(uy) &= (u' + \Delta u)(y' + \Delta y) - u'y' \\ &= y'\Delta u + u'\Delta y + \Delta u \cdot \Delta y.\end{aligned}$$

$$\therefore \frac{\Delta(uy)}{\Delta x} = y'\frac{\Delta u}{\Delta x} + u'\frac{\Delta y}{\Delta x} + \Delta u \cdot \frac{\Delta y}{\Delta x}.$$

$$\begin{aligned}\therefore \text{lt}\,\frac{\Delta(uy)}{\Delta x} &= \text{lt}\left(y'\frac{\Delta u}{\Delta x}\right) + \text{lt}\left[(u' + \Delta u)\frac{\Delta y}{\Delta x}\right] \\ &= y' \cdot \text{lt}\,(\Delta u/\Delta x) + \text{lt}\,(u' + \Delta u) \cdot \text{lt}\,(\Delta y/\Delta x).\end{aligned}$$

$$\therefore \frac{d(uy)}{dx} = y'\frac{du}{dx} + u'\frac{dy}{dx}. \qquad (1)$$

Since x' is any value of x, from (1) we have [7].

If in [7] we put for y the constant a, we obtain

$$d(au) = adu, \text{ or } [4].$$

If in [7] we put vw for u, we obtain

$$d(vwy) = wydv + vydw + vwdy, \text{ or } [8].$$

3. Prove $d(u/y) = (ydu - udy)/y^2$. [9]

By formula (1) in example 1, we have

$$\Delta\left(\frac{u}{y}\right) = \frac{u' + \Delta u}{y' + \Delta y} - \frac{u'}{y'} = \frac{y'\Delta u - u'\Delta y}{y'^2 + y'\Delta y}.$$

$$\therefore \frac{\Delta(u/y)}{\Delta x} = \frac{y'(\Delta u/\Delta x) - u'(\Delta y/\Delta x)}{y'^2 + y'\Delta y}.$$

Passing to limits, we obtain

$$\frac{d(u/y)}{dx} = \frac{y'(du/dx) - u'(dy/dx)}{y'^2}. \qquad (1)$$

Since x' is any value of x, from (1) we have [9].

4. Assuming the binomial theorem, prove $d(u^n) = nu^{n-1}du$. [13]

By formula (1) in example 1, we have

$$\Delta(u^n) = (u' + \Delta u)^n - u'^n = nu'^{n-1}\Delta u + \frac{n(n-1)}{\underline{|2}}u'^{n-2}\Delta u^2 + \cdots.$$

Dividing by Δx and passing to limits, we obtain [13].

5. Prove $d(u + y + z + a) = du + dy + dz$. [3]

6. If $y = z$, prove $dy = dz$. [1]

7. If a is constant, prove $da = 0$. [2]

56. By [33] of § 55 the problem of rates is reduced to one of limits. By theorems proved later in this chapter, proofs by limits are very much abbreviated.

The reader should note that we cannot write

$$\operatorname{lt}\frac{\Delta y}{\Delta x} \equiv \frac{\operatorname{lt}(\Delta y)}{\operatorname{lt}(\Delta x)} \equiv \frac{0}{0}. \qquad (1)$$

For any proof of the principle applied in (1) fails *when the limit of the divisor is zero.* Moreover, the determinate expression $\operatorname{lt}(\Delta y / \Delta x)$ cannot be identical with the indeterminate expression $0/0$.

To find $\operatorname{lt}(\Delta y / \Delta x)$, we find the limit of an equal variable, as in the examples of § 55; or we find the limit of some variable which, though not equal to $\Delta y / \Delta x$, has the same limit, as in § 63.

The theory of limits never gives rise to the form $a/0$, or to any of the indeterminate forms $0/0$, etc.

57. Derivative as a limit. By § 29 and [33] we have

$$\operatorname{lt}(\Delta y / \Delta x) \equiv \text{the derivative of } y \text{ with respect to } x.$$

Or, since $\Delta f(x) \equiv f(x + \Delta x) - f(x)$, we have

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] \equiv f'(x). \qquad (1)$$

$\operatorname{lt}(\Delta y / \Delta x)$, or dy/dx, is often denoted by $D_x y$.

58. Infinitesimals. *Zero* is defined by the identity

$$a - a \equiv 0.$$

An *infinitesimal* is a variable whose limit is zero.

Hence, $\Delta x \doteq 0$ may be read "Δx is an infinitesimal."

In approaching its limit zero, an infinitesimal becomes indefinitely small and continually smaller, but it never equals zero. Any small quantity *becomes* an infinitesimal when it *begins* to approach zero as its limit, not when it reaches any particular *degree of smallness.* A quantity, however small, which does not approach zero as its limit is not an infinitesimal.

Infinitesimals are indefinite variables used as auxiliaries in the study of finite quantities. Their *essence* and *utility* lie in their *having zero as their limit*, and not in their *smallness*. The reader should not make the study of them difficult and obscure by thinking of them as *mysteriously* small.

59. An **infinite** is a variable which under its law of change can exceed all assignable values, however great. The general symbol for an infinite is ∞. The reciprocal of an infinitesimal is an infinite, and conversely.

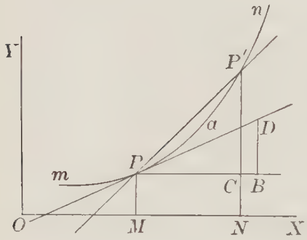
For example, when $x \doteq 0$, $1/x = \infty$; that is, when x is an infinitesimal, $1/x$ is an infinite.

When $\theta \doteq 0$, $\cot(\pm\theta) = \pm\infty$, and $\tan(\pi/2 \mp \theta) = \pm\infty$.

When $x \doteq 0$, $\log x = -\infty$.

An infinite does not approach a limit; in arithmetic value it increases without limit.

60. Geometric meaning of lt $(\Delta y/\Delta x)$. Let mn be the locus of $y = f(x)$, PP' a secant, and PD a tangent at P. Draw the ordinates MP and NP', also PC parallel to OX.



Let $OM = x$, and $MN = \Delta x$;
then $MP = y$, and $CP' = \Delta y$.

Hence, $\Delta y/\Delta x = CP'/PC =$ the slope of the secant PP'.

Conceive the secant PP' to revolve about P so that

$$\text{arc } PP' \doteq 0; \text{ then } \Delta x \doteq 0,\ \Delta y \doteq 0,$$

and the slope of the secant $\doteq$ the slope of the tangent at P.

Hence, lt $(\Delta y/\Delta x)$ = *the slope of the curve* $y = f(x)$ *at the point* (x, y).

Cor. If when $\Delta x \doteq 0$, $\Delta y/\Delta x$ varies, the locus of $y = f(x)$ is a curved line, and in general $\Delta y/\Delta x$ approaches a limit; but at a point where the locus is perpendicular to the x-axis $\Delta y/\Delta x = \infty$ when $\Delta x \doteq 0$.

61. *The limit of the ratio of an infinitesimal arc of any plane curve to its chord is unity.*

Let s represent the length of the arc mP (§ 60, fig.), and

$$\text{arc } PaP' = \Delta s; \text{ then } PC = \Delta x.$$

Since s is a function of x, we have

$$\operatorname{lt}(\Delta s / \Delta x) = ds / dx. \tag{1}$$

But $\operatorname{lt}(\text{chord } PP'/\Delta x) = \operatorname{lt}(\sec CPP')$

$$= \sec BPD = ds/dx. \tag{2}$$

Dividing (1) by (2), we obtain

$$\underset{\Delta s \doteq 0}{\text{limit}} \left[\frac{\Delta s}{\text{chord } PP'} \right] = 1. \tag{3}$$

Cor. Let u and s have the same meanings as in § 38; then

$$\underset{u \doteq 0}{\text{limit}} \left[\frac{u}{\sin u} \right] = \underset{s \doteq 0}{\text{limit}} \left[\frac{2s}{2 \sin u} \right] = \underset{XP \doteq 0}{\text{limit}} \left[\frac{2 \cdot XP}{2 \cdot CP} \right] = 1.$$

That is, *the limit of the ratio of an infinitesimal angle to its sine is unity.*

62. *The limit of the ratio of two variables is not changed when either is replaced by any other variable the limit of whose ratio to it is unity.*

Let α, α_1, β, and β_1 be any four variables, so related that

$$\operatorname{lt} \frac{\alpha}{\alpha_1} = 1, \quad \operatorname{lt} \frac{\beta}{\beta_1} = 1, \quad \text{and} \quad \operatorname{lt} \frac{\alpha}{\beta} = c. \tag{1}$$

$$\frac{\alpha}{\beta} = \frac{\alpha}{\beta} \cdot \frac{\alpha_1}{\alpha_1} \cdot \frac{\beta_1}{\beta_1} = \frac{\alpha_1}{\beta_1} \cdot \frac{\alpha}{\alpha_1} \cdot \frac{\beta_1}{\beta};$$

$$\therefore \operatorname{lt} \frac{\alpha}{\beta} = \operatorname{lt} \frac{\alpha_1}{\beta_1} \cdot \operatorname{lt} \frac{\alpha}{\alpha_1} \cdot \operatorname{lt} \frac{\beta_1}{\beta} = \operatorname{lt} \frac{\alpha_1}{\beta_1}, \qquad \text{by (1)}$$

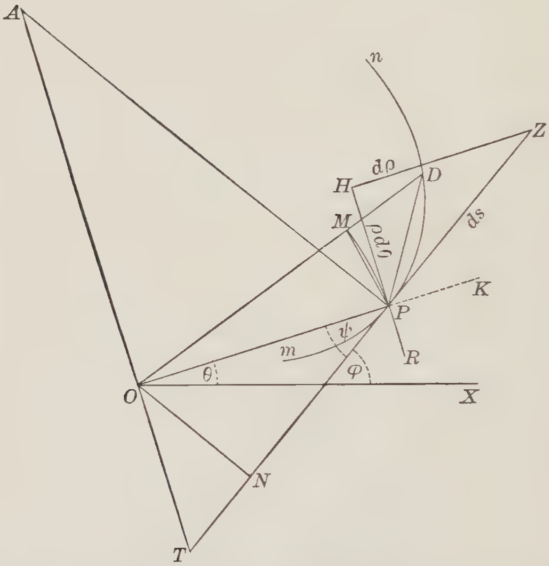
in which α is replaced by α_1, and β by β_1, without changing the limit.

This principle often enables us to simplify a problem of limits by substituting for an infinitesimal arc its chord, as in the following article.

63. Subtangent, subnormal, tangent, normal, in polar curves. Let arc $mP = s$, and arc $PD = \Delta s$; then $\angle POD = \Delta\theta$, circular arc $PM = \rho\Delta\theta$, and $MD = \Delta\rho$. Draw the chords PM and PD, the tangents RPH and TPZ, and ZH perpendicular to PH, Z being any point on the tangent PZ.

When $\Delta s \doteq 0$, the limiting positions of the secants PM and PD are the tangents RPH and TPZ, respectively; hence,

$$\begin{aligned} &\text{lt}\,(\angle PMD) = \angle RPK = \pi/2 = \angle PHZ, \\ &\text{lt}\,(\angle ODP) = \angle OPT = \psi \quad = \angle HZP, \\ \text{and}\quad &\text{lt}\,(\angle MPD) = \angle HPZ. \end{aligned}$$



Again, by §§ 61 and 62, and trigonometry, we have

$$\text{lt}\,\frac{\Delta\rho}{\Delta s} = \text{lt}\,\frac{MD}{\text{chord } PD} = \text{lt}\,\frac{\sin MPD}{\sin PMD};$$

$$\therefore \frac{d\rho}{ds} = \sin HPZ = \frac{HZ}{PZ}. \tag{1}$$

Also, $$\text{lt}\,\frac{\rho\Delta\theta}{\Delta s} = \text{lt}\,\frac{\text{chord } MP}{\text{chord } PD} = \text{lt}\,\frac{\sin MDP}{\sin PMD};$$

$$\therefore \frac{\rho d\theta}{ds} = \sin HZP = \frac{HP}{PZ}. \tag{2}$$

From (1) and (2), it follows that, if

$$ds = PZ, \quad d\rho = HZ \quad \text{and} \quad \rho d\theta = HP.$$

Draw OT perpendicular to OP, and PA and ON perpendicular to the tangent TP. Then the length PT is called the *polar tangent;* PA, the *polar normal;* OA, the *polar subnormal;* and OT, the *polar subtangent.*

From the right-angled triangle HPZ we have

$$ds^2 = d\rho^2 + \rho^2 d\theta^2; \tag{3}$$

$$\sin\psi = \frac{\rho d\theta}{ds}, \quad \cos\psi = \frac{d\rho}{ds}, \quad \tan\psi = \frac{\rho d\theta}{d\rho}. \tag{4}$$

$$\text{Polar subt.} = OT = OP\tan\psi = \rho^2 d\theta / d\rho. \tag{5}$$

$$\text{Polar subn.} = OA = OP\cot\psi = d\rho / d\theta. \tag{6}$$

$$\text{Polar tan.} = PT = \sqrt{OP^2 + OT^2} = \rho\sqrt{1 + \rho^2\frac{d\theta^2}{d\rho^2}}. \tag{7}$$

$$\text{Polar norm.} = AP = \sqrt{OP^2 + OA^2} = \sqrt{\rho^2 + \frac{d\rho^2}{d\theta^2}}. \tag{8}$$

$$\begin{aligned} p = ON = OP\sin\psi &= \rho^2 d\theta / ds \\ &= \frac{\rho^2}{\sqrt{\rho^2 + (d\rho / d\theta)^2}}. \end{aligned} \tag{9}$$

$$\phi = \psi + \theta. \tag{10}$$

Cor. If PZ represents the velocity at P of (ρ, θ) along the line of its path, HZ and PH will represent its component velocities at P along the radius vector and a line perpendicular to it.

EXAMPLES.

1. Find the subtangent, subnormal, tangent, normal, and p of the spiral of Archimedes $\rho = a\theta$.

Ans. subn. $= a$; subt. $= \rho^2/a$; norm. $= \sqrt{\rho^2 + a^2}$;
tan. $= \rho\sqrt{1 + \rho^2/a^2}$; $p = \rho^2/(\rho^2 + a^2)^{1/2}$.

2. In the spiral of Archimedes show that $\tan\psi = \theta$.

3. Find the subtangent, subnormal, tangent, and normal of the logarithmic spiral $\rho = a^\theta$.

Ans. subt. $= \rho/\log a$; subn. $= \rho\log a$;
tan. $= \rho\sqrt{1 + (\log a)^{-2}}$; norm. $= \rho\sqrt{1 + (\log a)^2}$.

4. In the logarithmic spiral, show that ψ is constant.

If $a = e$, $\psi = \pi/4$, subt. = subn., and tan. = norm.

Since the logarithmic spiral cuts every radius vector at the same angle, it is often called the *equiangular* spiral.

5. Find the subtangent, subnormal, and p of the lemniscate of Bernouilli $\rho^2 = a^2\cos 2\theta$.

Ans. subt. $= -\rho^3/a^2\sin 2\theta$; subn. $= -a^2\sin 2\theta/\rho$.
$p = \rho^3/\sqrt{\rho^4 + a^4\sin^2 2\theta} = \rho^3/a^2$.

6. In the lemniscate show that $\psi = 2\theta + \pi/2$ and $\phi = 3\theta + \pi/2$.

7. In the curve $\rho = a\sin^3(\theta/3)$, show that $\phi = 4\psi$. § 155, fig. 14.

8. In the parabola $\rho = a\sec^2(\theta/2)$, show that $\phi + \psi = \pi$.

9. The velocity of a point along the spiral $\rho = a\theta$ is v; find its component velocities along the radius vector and a line perpendicular to it.

10. By the method of limits (§ 60, fig.) prove that, if $PB = dx$, $BD = dy$ and $PD = ds$, where $s = mP$.

64. Orders of infinitesimals. Any variable which is neither an infinitesimal nor an infinite is called a **finite** variable.

Two infinitesimals are said to be of the *same* order when their ratio is a constant or a *finite* variable.

For example, when $\Delta x \doteq 0$, $7\,\Delta x$ and $(5\,a + 6)\Delta x$ are infinitesimals of the same order; so also are $9\,(\Delta x)^2$ and $(8\,x - 7)\,(\Delta x)^2$; so also are Δy and Δx when $\text{lt}\,(\Delta y \,/\, \Delta x)$ is other than zero. Again, when $\theta \doteq \pi/2$, $1 - \sin\theta$ and $\cos^2\theta$ are infinitesimals of the same order; for

$$\underset{\theta \doteq \pi/2}{\text{limit}}\left[\frac{1 - \sin\theta}{\cos^2\theta}\right] = \underset{\theta \doteq \pi/2}{\text{limit}}\left[\frac{1}{1 + \sin\theta}\right] = \frac{1}{2}.$$

In order to classify the infinitesimals in any problem as belonging to different orders, we choose some one of them as the *principal infinitesimal*, and adopt the following definitions:

An infinitesimal is of the *first order* when it is of the same order as the principal infinitesimal; of the *second order* when it is of the same order as the *square* of the principal infinitesimal; and so on.

In general, an infinitesimal is of the nth *order* when it is of the same order as the nth *power* of the principal infinitesimal.

Thus, when Δx is taken as the principal infinitesimal and $\text{lt}\,(\Delta y \,/\, \Delta x)$ is other than zero, Δy is an infinitesimal of the *first* order; $5\,x\,(\Delta x)^2$ and $7\,y\Delta y\Delta x$ are infinitesimals of the *second* order; $7\,y\,(\Delta x)^3$ and $4\,x\,(\Delta x)^2\Delta y$ are of the *third* order; and $y(\Delta x)^n$ and $x^2(\Delta x)^{n-m}\,(\Delta y)^m$ are of the nth order.

65. Notation. Let $v_1, v_2, \cdots, v_n$ represent finite variables or any constants except zero, and let i represent the principal infinitesimal; then $v_1 i$, $v_2 i^2, \cdots, v_n i^n$ will represent respectively infinitesimals of the first, the second, $\cdots$, the nth order.

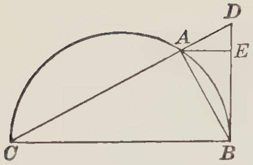
According to this notation, $\Delta x = i$ is read "Δx is the principal infinitesimal"; $5\,x\Delta x = vi$ is read "$5\,x\Delta x$ is an infinitesimal of the first order"; $7\,x\Delta x\Delta y = v_2 i^2$ is read "$7\,x\Delta x\Delta y$ is an infinitesimal of the second order"; and so on. The subscript need not be written with the first v which appears in any problem or discussion.

A *finite* quantity may be regarded as an infinitesimal of the *zero* order; for $x \equiv xi^0$ and $vi^0 \equiv v$.

Ex. If $\theta = i$, $1 - \cos\theta = vi^2$.

For $$\underset{\theta \doteq 0}{\text{limit}}\left[\frac{1 - \cos\theta}{\theta^2}\right] = \underset{\theta \doteq 0}{\text{limit}}\left[\frac{1 - \cos\theta}{\sin^2\theta}\right] \qquad §§\ 61, 62$$

$$= \underset{\theta \doteq 0}{\text{limit}}\left[\frac{1}{1 + \cos\theta}\right] = \frac{1}{2}. \qquad §\ 64$$



66. Geometric illustration of infinitesimals of different orders.

Let CAB be a right angle inscribed in the semicircle CAB, BD a tangent at B, and AE a perpendicular to BD. From the similar triangles CAB, BAD, and AED we have

$$AD : AB = AB : AC; \quad \therefore AD = (1/\overline{AC})\,\overline{AB}^2; \tag{1}$$

and

$$DE : AD = AB : BC; \quad \therefore DE = (1/\overline{BC}) \cdot \overline{AD} \cdot \overline{AB}. \tag{2}$$

Suppose A to approach B, and let $AB = i$; then, from (1) and (2),

$$AD = (1/\overline{AC}) \cdot i^2 = vi^2,$$

and

$$DE = (1/\overline{BC}) \cdot vi^2 \cdot i = v_3 i^3.$$

67. Orders of products and quotients.

The order of the product of two or more infinitesimals is equal to the sum of the orders of the factors.

For
$$vi^n \cdot v_m i^m = vv_m i^{n+m}.$$

The order of the quotient of any two infinitesimals is equal to the order of the dividend minus the order of the divisor.

For
$$vi^n / v_m i^m = (v/v_m)\, i^{n-m}.$$

68. *If the ratio of one infinitesimal to another is infinitesimal, the first is of a higher order than the second; and conversely.*

For
$$vi^n / v_m i^m \equiv (v/v_m)\, i^{n-m} \doteq 0,$$

when, and only when, $n > m$.

Cor. If the ratio of one infinitesimal to another is infinite, the first is of a lower order than the second.

69. *If the limit of the ratio of two variables is unity, their difference is an infinitesimal of a higher order than either; and conversely.*

Let $$a = \beta + \epsilon;$$

then $$\operatorname{lt}(a/\beta) = 1 + \operatorname{lt}(\epsilon/\beta).$$

If $$\operatorname{lt}(a/\beta) = 1,\ \operatorname{lt}(\epsilon/\beta) = 0;$$

hence, by § 68, ϵ is of a higher order than β.

Conversely, if ϵ is of a higher order than β,

$$\operatorname{lt}(\epsilon/\beta) = 0;\ \therefore \operatorname{lt}(a/\beta) = 1.$$

Cor. If arc $PP' = i$, arc $PP' =$ chord $PP' + vi^n$, where $n > 1$.

Also, if angle $u = i$, $u = \sin u + vi^n$, where $n > 1$. § 61

70. *From sums of infinitesimals of different orders, all infinitesimals of the higher orders vanish in the limit of a ratio.*

For $$\operatorname{lt}\frac{vi + v_2i^2 + v_3i^3 + \cdots}{v'i + v'_2i^2 + v'_3i^3 + \cdots} = \operatorname{lt}\frac{vi}{v'i}. \qquad \text{§ 62}$$

This principle of limits often greatly shortens the operation of finding the limit of a ratio, and together with [33] of § 55 furnishes the following simple

71. Rule for differentiating a function.

(1) *Find the value of the increment of the function in terms of the increments of its variables.*

(2) *Supposing the increments to be infinitesimals of the first order, in all sums drop the infinitesimals of the higher orders, and in the remaining terms substitute differentials for increments.*

For by § 70 the infinitesimals of the higher orders in a sum will vanish in the limit, and by [33] of § 55 differentials will take the place of increments in the remaining terms.

Cor. Anticipating, in step (1), the result of step (2) we need to express exactly only those terms of $\Delta f(u)$ which are linear in Δu. (See examples 7–11.)

EXAMPLES.

$u, y, \cdots$ being different functions of x, by § 71 prove

1. $d(uy) = ydu + udy$, or [7].

$$\Delta(uy) = (u + \Delta u)(y + \Delta y) - uy \qquad \text{§ 8, example 3}$$
$$= y\Delta u + u\Delta y + \Delta u \Delta y.$$

Let $\Delta x = i$; then $\Delta u \Delta y = vi^2$. Hence, dropping $\Delta u \Delta y$, and substituting $d(uy)$, du, and dy, respectively, for $\Delta(uy)$, Δu, and Δy in the remaining terms we obtain [7].

2. $d(u/y) = (ydu - udy)/y^2$, or [9].

$$\Delta\left(\frac{u}{y}\right) = \frac{u + \Delta u}{y + \Delta y} - \frac{u}{y} = \frac{y\Delta u - u\Delta y}{y^2 + y\Delta y}.$$

$y^2 = vi^0$, and $y\Delta y = v_1 i$; hence, in the sum $y^2 + y\Delta y$, by (2) of § 71, we drop $y\Delta y$. Substituting differentials for increments in the remaining terms, we obtain [9].

3. $d(au) = adu$, or [4].

4. Formula [3].

5. Formula [8].

6. $d(u^n) = nu^{n-1}du$, or [13].

$$\Delta(u^n) = (u + \Delta u)^n - u^n$$
$$= nu^{n-1}\Delta u + \frac{n(n-1)}{\underline{|2}} u^{n-2}\Delta u^2 + \cdots.$$
$$\therefore d(u^n) = nu^{n-1}du.$$

7. $d \sin u = \cos u du$, or [17].

$$\Delta(\sin u) = \sin(u + \Delta u) - \sin u$$
$$= \cos u \sin \Delta u + \sin u \cos \Delta u - \sin u \qquad \text{by Trig.}$$
$$= \cos u (\Delta u - vi^n) - (1 - \cos \Delta u) \sin u. \qquad \text{§ 69, Cor.}$$
$$= \cos u \cdot \Delta u - vi^n \cdot \cos u - v_2 i^2 \sin u. \qquad \text{§ 65, example}$$
$$\therefore d \sin u = \cos u du.$$

In obtaining the value of $\Delta(\sin u)$ we express *exactly* only those terms which are linear in Δu; for by § 70 all the other terms vanish in the limit.

8. $d \cos u = -\sin u du$, or [18].

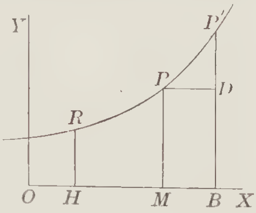
9. $ds = \sqrt{dx^2 + dy^2}$.

Let $\Delta s = \text{arc } PP' = i$; § 60, fig.

then $\Delta s = \text{chord } PP' + vi^n$, where $n > 1$, § 69, Cor.

$$= \sqrt{\Delta x^2 + \Delta y^2} + vi^n.$$
$$\therefore ds = \sqrt{dx^2 + dy^2}.$$

10. Find the differential of the area between the x-axis, the curve RP, or $y = f(x)$, the fixed ordinate HR, and the variable ordinate MP, or y.



Let P be any point (x, y) on the curve.

Conceive the area $HRPM$ as generated by the ordinate MP, or y, and denote this area by A.

Let $MB = \Delta x$; then $\Delta y = DP'$,

and $\Delta A = MBP'P = y\Delta x + PDP'$.

Let $\Delta x = i$; then, since

$$PDP' < \Delta x \Delta y, \quad PDP' = vi^2.$$

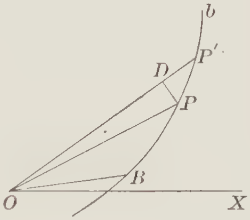
Hence, $$\Delta A = y\Delta x + vi^2;$$

$$\therefore dA = ydx. \tag{1}$$

If $$dx = MB, \qquad dA = MBDP.$$

From (1) the x-rate of A is y to 1 of x.

11. Find the differential of the area of a polar curve.



Let OB be any fixed radius vector, and P any point (ρ, θ) on the curve BPb referred to the pole O and the polar axis OX. Conceive the area BOP as generated by the rotation of the radius vector ρ, and denote it by A.

Let $\angle POP' = \Delta\theta$, and draw the circular arc PD with O as a centre; then

$$\Delta\rho = DP', \qquad \rho\Delta\theta = PD,$$

and $$\Delta A = OPP' = OPD + DPP'$$
$$= \tfrac{1}{2}\rho^2\Delta\theta + DPP'.$$

Let $\Delta\theta = i$; then, since $DP'P < \rho\Delta\theta \cdot \Delta\rho$,

$$DPP' = vi^2.$$

Hence, $$\Delta A = \tfrac{1}{2}\rho^2\Delta\theta + vi^2;$$

$$\therefore dA = \tfrac{1}{2}\rho^2 d\theta. \tag{1}$$

If $d\theta = \angle POP'$, $dA =$ the circular sector OPD.

From (1) the θ-rate of A is $\rho^2/2$ to 1 of θ.

12. Prove the theorem in § 70 by § 53.

13. If an infinitesimal be multiplied or divided by any finite quantity, the order of the infinitesimal is not changed.

72. Orders of infinites. The reciprocal of an infinitesimal of any positive order is an infinite of the same order; hence, the different positive orders of infinites may be regarded as negative orders of infinitesimals; and conversely.

According to the notation in § 65 we may write the different positive and negative orders of infinitesimals as below:

$$v_{-n}i^{-n},\ \cdots,\ v_{-2}i^{-2},\ v_{-1}i^{-1},\ vi^{0},\ v_1i^{1},\ v_2i^{2},\ \cdots\ v_ni^{n}, \qquad (1)$$

where the negative orders of infinitesimals are positive orders of infinites.

Let $\infty = i^{-1}$; then the series in (1) becomes

$$v_{-n}\infty^{n},\ \cdots,\ v_{-2}\infty^{2},\ v_{-1}\infty,\ v\infty^{0},\ v_1\infty^{-1},\ v_2\infty^{-2},\ \cdots,\ v_n\infty^{-n},$$

where the negative orders of infinites are positive orders of infinitesimals.

Infinitesimals and infinites of all orders are variables whose general laws of combination are the same as those of finite quantities.

An infinite or an infinitesimal of the zero order is a finite quantity.

COR. The product of an infinitesimal and an infinite of the same order is a finite quantity; for

$$v_{-n}i^{-n} \cdot v_ni^{n} \equiv v_{-n}v_ni^{0}.$$

When i is indefinitely small each term in (1) is indefinitely small in comparison with the one which precedes it, but indefinitely large compared with the one which follows it.

73. $\infty^{n} \cdot 0 \equiv 0$, *where n is any positive number.*

Let
$$\infty i^{m} \equiv v, \qquad (1)$$
where neither v nor its limit is zero, and m is any positive number.

Since $i^{mn} \cdot 0 \equiv 0$, it evidently follows that

$$\begin{aligned}\infty^{n} \cdot 0 \equiv \infty^{n}\,(i^{mn}\,0) &\equiv (\infty i^{m})^{n} \cdot 0 \\ &\equiv v^{n} \cdot 0 \equiv 0. \qquad \text{by (1)}\end{aligned}$$

By Cor. of § 72, ∞ in (1) denotes an infinite of the mth order.

COR. $\infty^0 \equiv 1$ *and* $1^{\pm\infty} \equiv 1$.

For
$$\begin{aligned}\log(\infty^0) &\equiv 0 \log \infty\\ &\equiv 0 \cdot \infty \equiv 0;\\ \therefore \infty^0 &\equiv 1.\end{aligned}$$

Again
$$\begin{aligned}\log(1^{\pm\infty}) &\equiv \pm\infty \log 1\\ &\equiv \pm\infty \cdot 0 \equiv 0;\\ \therefore 1^{\pm\infty} &\equiv 1.\end{aligned}$$

74. ap, **or absolute infinity.** The expression $a/0$ frequently occurs in mathematics. The question arises, What does the expression $a/0$ (written as one symbol ap, which is read 'a-by-zero') symbolize?

Any power of an infinite expresses no part of $a/0$ as a quotient; for, by § 73, ∞^n into the divisor 0 equals 0, or no part of the dividend a. Since ap symbolizes that of which no part can be expressed by any power of a mathematical infinite, it must symbolize that which transcends all mathematical quantity, or *absolute infinity*, of which we can have no positive idea.

Since ap is not a mathematical quantity, it is not subject to mathematical laws, and the expressions

$$ap/ap,\ ap \cdot 0,\ ap - ap,\ ap^0,\ 1^{ap}$$

are indeterminate forms. See Chapter V.

We would naturally conclude that

$$2/0 = 2\,(1/0),\ 3/0 = 3\,(1/0),\ \cdots.$$

But
$$\tan(\pi/2) = 1/0,\ \text{or } 2/0,\ \text{or } 3/0,\ \cdots.$$

The inconsistency of these results illustrates the impossibility of reasoning with the symbol ap.

Sometimes when one of two related variables assumes the form ap, we know the value of the other.

For example, when $\tan\phi = ap$, *i.e.* when $\tan\phi$ assumes the form ap, we know that ϕ is coterminal with $\pi/2$, or $3\pi/2$; and conversely.

When $\cot\phi = ap$, we know that ϕ is coterminal with 0 or π.

If, when $x = c$, $f(x) = 0/a = 0$, the reciprocal of $f(x)$ assumes the form $a/0$, or ap, when $x = c$.

75. In this chapter, to obtain the ratio of differentials, or the ratio of rates, we employ infinitesimal increments of variables as *auxiliary* quantities. The division of infinitesimals into orders affords clear and brief statements of principles of limits which greatly abridge and simplify the work of finding the ratio of differentials. In this as in the previous chapters, differentials are regarded as finite quantities.

76. Limit in position. When according to its law of motion a point, or line, approaches indefinitely near and continually nearer a fixed point, or line, but can never reach it, the fixed point, or line, is called the *limit* of the variable point, or line.

For example, when in § 60 arc $PP' \doteq 0$, the fixed point P is the limit of P', and the tangent PD is the limit of the secant PP'.

Whether the word limit has reference to magnitude or to position will always be evident from the context.

EXAMPLES.

1. When $x \doteq c$, $a/(x-c) = \pm\infty$; when $x = c$, $a/(x-c) = ap$.

2. When $x \doteq 0$, $a/x = \pm\infty$; when $x = 0$, $a/x = ap$.

3. When $\phi \doteq \pi/2$, $\sec\phi = \pm\infty$; when $\phi = \pi/2$, $\sec\phi = ap$.

4. When $\phi \doteq 0$, $\csc\phi = \pm\infty$; when $\phi = 0$, $\csc\phi = ap$.

5.
$$\frac{a/\phi(x)}{a/f(x)} \equiv \frac{f(x)}{\phi(x)}, \tag{1}$$
and
$$f(x)\cdot[a/\phi(x)] \equiv af(x)/\phi(x). \tag{2}$$

If $f(c) \equiv \phi(c) \equiv 0$, (1) and (2) become respectively
$$ap/ap \equiv 0/0, \text{ and } 0 \cdot ap \equiv 0/0.$$

Hence, any expression in x which assumes the indeterminate form ap/ap or $0 \cdot ap$ for any value of x can be so transformed as to assume the form $0/0$ for the same value of x.

6. When $\Delta x \doteq 0$, for what points on the locus of $y = f(x)$ is $\Delta y/\Delta x$ infinitesimal? infinite? finite?

CHAPTER IV.

SUCCESSIVE DIFFERENTIATION.

77. Successive differentials. The differential of du is called the *second* differential of u; the differential of the second differential of u is called the *third* differential of u; and so on. $d\,(du)$ is written d^2u; $d\,(d^2u)$, or $d\,d\,du$, is written d^3u; and so on. The figure above d denotes how many times in succession the operation of differentiation has been performed. $du, d^2u, d^3u, \cdots, d^nu$ are called the *successive differentials* of u.

The differential of an independent variable, being arbitrary, is supposed to have the same size at all values of the variable.

Hence, when (as in this chapter) x is independent, dx is to be treated as a constant in successive differentiation.

Ex. Find the successive differentials of u when $u = ax^4$.

$$\begin{aligned} du &= 4\,ax^3dx\,; \\ d^2u &= 4\,adx \cdot d\,(x^3) = 12\,ax^2dx^2; \\ d^3u &= 12\,adx^2 \cdot d\,(x^2) = 24\,axdx^3; \\ d^4u &= 24\,adx^4;\ d^5u = 0. \end{aligned}$$

Note that $d^2u \equiv d\,du$; $du^2 \equiv (du)^2$; $d\,(u^2) \equiv 2\,udu$.

EXAMPLES.

Find du, d^2u, and d^3u, when

1. $u = 5\,x^3 + 2\,x^2 - 3\,x$. $\qquad d^3u = 30\,dx^3$.
2. $u = (x^2 - 6\,x + 12)\,e^x$. $\qquad d^3u = x^2e^xdx^3$.
3. $u = x^2 \log x$. $\qquad d^3u = 2\,x^{-1}\,dx^3$.
4. $u = \log \sin x$. $\qquad d^3u = 2\cos x \sin^{-3} x dx^3$.

5. $u = \tan x$. $d^3u = (6 \sec^4 x - 4 \sec^2 x)\, dx^3$.

6. $y = \log ax$; find d^4y. $d^4y = -6x^{-4}\, dx^4$.

7. $y = e^{-x} \cos x$; find d^4y. $d^4y = -4e^{-x} \cos x dx^4$.

8. $y = e^x \sin x$; find d^6y. $d^6y = -8e^x \cos x dx^6$.

78. Successive derivatives. The derivative of the first derivative of a function is called the *second derivative* of the function; the derivative of the second derivative is called the *third derivative;* and so on.

When x is independent,

$$\frac{d}{dx}\frac{du}{dx} \equiv \frac{d^2u}{dx^2}, \quad \frac{d}{dx}\frac{d^2u}{dx^2} \equiv \frac{d^3u}{dx^3}; \quad \cdots, \quad \frac{d}{dx}\frac{d^{n-1}u}{dx^{n-1}} \equiv \frac{d^nu}{dx^n}.$$

Dividing by dx^3 both members of the answers to examples 1–5 in § 77, we obtain in each case the *third derivative* of u with respect to x.

The successive derivatives of $f(x)$ are denoted by

$$f'(x),\ f''(x),\ f'''(x),\ f^{\text{IV}}(x),\ \cdots,\ f^n(x).$$

Thus if
$$f(x) = x^4, \quad f'(x) = 4x^3,\ f''(x) = 12x^2,$$
$$f'''(x) = 24x,\ f^{\text{IV}}(x) = 24, \quad f^{\text{V}}(x) = 0.$$

Hence, if $u = f(x)$ and x is independent,

$$\frac{du}{dx} = f'(x),\ \frac{d^2u}{dx^2} = f''(x),\ \cdots,\ \frac{d^nu}{dx^n} = f^n(x).$$

Successive derivatives are often called *successive differential coefficients.*

79. The **nth derivatives** of some functions can be readily obtained by inspection.

Ex. 1. $f(x) = e^x$; find $f^n(x)$.

$$f'(x) = e^x,\ f''(x) = e^x,\ f'''(x) = e^x,\ \cdots; \quad \therefore f^n(x) = e^x.$$

Ex. 2. $f(x) = a^x$; find $f^n(x)$.

$$f'(x) = \log a \cdot a^x,\ f''(x) = (\log a)^2 a^x,\ f'''(x) = (\log a)^3 a^x,\ \cdots;$$
$$\therefore f^n(x) = (\log a)^n a^x.$$

Ex. 3. $f(x) = \log(1 + x)$; find $f^n(x)$.

$$f'(x) = (1 + x)^{-1},\ f''(x) = (-1)(1 + x)^{-2},$$
$$f'''(x) = (-1)^2 \lfloor\underline{2}\,(1 + x)^{-3},\ f^{\text{IV}}(x) = (-1)^3 \lfloor\underline{3}\,(1 + x)^{-4},\ \cdots;$$
$$\therefore f^n(x) = (-1)^{n-1} \lfloor\underline{n-1}\,(1 + x)^{-n}.$$

Ex. 4. $f(\theta) = \cos a\theta$; find $f^n(\theta)$.

$$f'(\theta) = -a \sin a\theta = a \cos(a\theta + \pi/2),$$
$$f''(\theta) = -a^2 \sin(a\theta + \pi/2) = a^2 \cos(a\theta + 2 \cdot \pi/2),$$
$$f'''(\theta) = -a^3 \sin(a\theta + \pi) = a^3 \cos(a\theta + 3 \cdot \pi/2),$$
$$\cdots\cdots\cdots\cdots$$
$$\therefore f^n(\theta) = a^n \cos(a\theta + n \cdot \pi/2).$$

80. *Each of the successive derivatives of* f(x) *equals the* x-*rate of the preceding derivative.*

For $\qquad f^n(x) = df^{n-1}(x)/dx =$ the x-rate of $f^{n-1}(x)$.

Cor. $f^{n-1}(x)$ is an increasing or a decreasing function of x according as $f^n(x)$ is positive or negative; and conversely.

EXAMPLES.

1. $f(x) = cx^3 + ax^2 + a$. $\qquad f'''(x) = 6c,\ f^{\text{IV}}(x) = 0$.

   Here $f''(x)$ changes at the rate of $6c$ to 1 of x (§ 80).

2. $f(x) = x^6 + 4x^4 + 3x + 2$. $\qquad f^{\text{VI}}(x) = \lfloor\underline{6}$.

3. $f(x) = x \log x$. $\qquad f^n(x) = (-1)^{n-2} \lfloor\underline{n-2}\, x^{1-n}$.

   Here $f^{n-1}(x)$ changes at the rate of $(-1)^{n-2} \lfloor\underline{n-2}\, x^{1-n}$ to 1 of x.

4. $f(x) = ax^m$. $\qquad f^n(x) = am(m-1)(m-2)\cdots(m-n+1)x^{m-n}$.

5. $f(x) = x^3 \log x$. $\qquad f^{\text{IV}}(x) = 6x^{-1}$.

6. $y = \log(e^x + e^{-x})$. $\qquad \dfrac{d^3y}{dx^3} = -8\dfrac{e^x - e^{-x}}{(e^x + e^{-x})^3}$.

7. $f(x) = \frac{1}{6}x^3(\log x - 5/6)$. $\qquad f^n(x) = (-1)^{n-4} \lfloor\underline{n-4}\, x^{3-n}$.

8. $f(x) = (x^2 - 3x + 3)e^{2x}$. $\qquad f'''(x) = 8x^2e^{2x}$.

9. $f(x) = x^4 \log x$. $\qquad f^{\text{VI}}(x) = -\lfloor\underline{4}\, x^{-2}$.

10. $f(x) = x^x.$ $\qquad f''(x) = x^x(1 + \log x)^2 + x^{x-1}.$

11. $y = \dfrac{x^3}{1 - x}.$ $\qquad \dfrac{d^4y}{dx^4} = \dfrac{24}{(1 - x)^5}.$

12. $f(x) = e^{ax}.$ $\qquad f^n(x) = a^n e^{ax}.$

13. $f(\theta) = \sin a\theta.$ $\qquad f^n(\theta) = a^n \sin(a\theta + n \cdot \pi/2).$

14. $f(x) = (1 + x)^m.$ $\qquad f^n(x) = m(m - 1)\cdots(m - n + 1)(1 + x)^{m-n}.$

15. $y = \dfrac{1}{4x + 2} = (4x + 2)^{-1}.$ $\qquad \dfrac{d^ny}{dx^n} = \dfrac{(-1)^n 4^n \underline{|n}}{(4x + 2)^{n+1}}.$

16. $y = \dfrac{2 - x}{2 + x}.$ $\qquad \dfrac{d^ny}{dx^n} = \dfrac{(-1)^n 4 \underline{|n}}{(2 + x)^{n+1}}.$

$$\frac{2 - x}{2 + x} = -1 + \frac{4}{2 + x} = -1 + 4(2 + x)^{-1}.$$

17. $y = \dfrac{6x - 1}{3x + 2}.$ $\qquad \dfrac{d^ny}{dx^n} = \dfrac{-5(-1)^n 3^n \underline{|n}}{(3x + 2)^{n+1}}.$

18. $y = \dfrac{2}{4x^2 - 1}.$ $\qquad \dfrac{d^ny}{dx^n} = (-1)^n 2^n \underline{|n} \left\{ \dfrac{1}{(2x - 1)^{n+1}} - \dfrac{1}{(2x + 1)^{n+1}} \right\}.$

$$\frac{2}{4x^2 - 1} = (2x - 1)^{-1} - (2x + 1)^{-1}.$$

Prove each of the following differential equations:

19. When $u = \sqrt{\sec 2x}$, $d^2u/dx^2 = 3u^5 - u$.

$$\frac{du}{dx} = \frac{\sec 2x \tan 2x}{\sqrt{\sec 2x}} = u \tan 2x. \tag{1}$$

$$\therefore \frac{d^2u}{dx^2} = \tan 2x \frac{du}{dx} + 2u \sec^2 2x.$$

$$= u \tan^2 2x + 2u \sec^2 2x. \qquad \text{by (1)}$$

$$= u(\sec^2 2x - 1) + 2u \sec^2 2x = 3u^5 - u.$$

20. When $u = e^x \sin x$, $\dfrac{d^2u}{dx^2} - 2\dfrac{du}{dx} + 2u = 0.$

21. When $u = a \sin(\log x)$, $x^2 \dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + u = 0.$

22. When $y = e^{cx} \sin mx$, $\dfrac{d^2y}{dx^2} - 2c\dfrac{dy}{dx} + (c^2 + m^2)\,y = 0.$

23. When $y = \sin(\sin x)$, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\tan x + y\cos^2 x = 0.$

24. When $u = \cos(a \sin^{-1} x)$, $(1 - x^2)\dfrac{d^2u}{dx^2} - x\dfrac{du}{dx} + a^2u = 0.$

25. When $u = (\sin^{-1} x)^2$, $(1 - x^2)\dfrac{d^2u}{dx^2} - x\dfrac{du}{dx} = 2.$

26. When $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$, $\dfrac{d^2y}{dx^2} = \dfrac{y}{a^2}.$

Of each of the following *implicit* functions obtain that derivative which is given at its right:

27. If $y^2 = 2xy - c$, $d^2y/dx^2 = c/(x - y)^3$.

$$d(y^2) = d(2xy - c); \quad \therefore \frac{dy}{dx} = \frac{y}{y - x}. \qquad (1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(y - x)\dfrac{dy}{dx} - y\left(\dfrac{dy}{dx} - 1\right)}{(y - x)^2} = \frac{y - x\dfrac{dy}{dx}}{(y - x)^2}. \qquad (2)$$

From (1), (2), and the given equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{y^2 - 2xy}{(y - x)^3} = \frac{c}{(x - y)^3}.$$

28. If $y^2 = 4px$, $d^3y/dx^3 = 24p^3/y^5$.

$$dy/dx = 2p/y;$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2p(dy/dx)}{y^2} = -\frac{4p^2}{y^3}.$$

$$\therefore \frac{d^3y}{dx^3} = \frac{12y^2p^2(dy/dx)}{y^6} = \frac{24p^3}{y^5}.$$

29. If $x^2 + y^2 = r^2$, $d^2y/dx^2 = -r^2/y^3$.

30. If $y^3 = a^2x$, $d^2y/dx^2 = -2a^4/9y^5$.

31. If $x^2/a^2 + y^2/b^2 = 1$, $d^2y/dx^2 = -b^4/a^2y^3$.

32. If $x^2/a^2 - y^2/b^2 = 1$, $d^2y/dx^2 = -b^4/a^2y^3$.

33. If $y^2 = \dfrac{x^3}{2a - x}$, $\dfrac{d^2y}{dx^2} = \pm \dfrac{3a^2}{\sqrt{x(2a - x)^5}}$.

34. If $x^{2/3} + y^{2/3} = a^{2/3}$, $d^2y/dx^2 = a^{2/3}/3y^{1/3}x^{4/3}$.

35. If $y^2 - 2axy + x^2 = c$,

$$\frac{d^2y}{dx^2} = \frac{(a^2-1)(y^2 - 2axy + x^2)}{(y - ax)^3} = \frac{c(a^2-1)}{(y-ax)^3}.$$

36. If $y^3 + x^3 = 3axy$, $d^2y/dx^2 = -2a^3xy/(y^2 - ax)^3$.

37. If $e^{x+y} = xy$, $\dfrac{d^2y}{dx^2} = -\dfrac{y[(x-1)^2 + (y-1)^2]}{x^2(y-1)^3}$.

81. Leibnitz's theorem is a formula for the nth differential of the product of two variables.

Let u and v be functions of x; then

$$d(uv) = du \cdot v + u\,dv. \tag{1}$$

In general, du and dv will be functions of x; hence,

$$\begin{aligned} d^2(uv) &= d^2u \cdot v + du\,dv + du\,dv + u\,d^2v \\ &= d^2u \cdot v + 2\,du\,dv + u\,d^2v. \end{aligned} \tag{2}$$

$$\therefore d^3(uv) = d^3u \cdot v + 3\,d^2u\,dv + 3\,du\,d^2v + u\,d^3v. \tag{3}$$

The coefficients and the *exponents of operation* in (2) and (3) follow the laws of the coefficients and exponents in the Binomial Theorem. However far we continue the differentiation, these laws will evidently hold; hence, we have

$$\begin{aligned} d^n(uv) = d^nu \cdot v + nd^{n-1}u\,dv + \frac{n(n-1)}{\lfloor\!\underline{2}}\,d^{n-2}u\,d^2v + \cdots \\ + n\,du\,d^{n-1}v + u\,d^nv. \end{aligned} \tag{4}$$

EXAMPLES.

1. Find $d^5(e^{ax}x^2)$.

Here $u = e^{ax}, \quad d^n u = a^n e^{ax}\, dx^n;$

and $v = x^2, \quad dv = 2\,x\,dx, \quad d^2v = 2\,dx^2, \quad d^3v = 0.$

Substituting these values in (4), we obtain, when $n = 5$,

$$d^5(e^{ax}x^2) = (a^5e^{ax}\cdot x^2 + 5\cdot a^4e^{ax}\cdot 2\,x + 10\cdot a^3e^{ax}\cdot 2)\,dx^5$$
$$= a^3e^{ax}(a^2x^2 + 10\,ax + 20)\,dx^5.$$

2. Find $d^n(x^2 \sin ax)$.

Here $u = \sin ax, \quad d^n u = a^n \sin(ax + n\cdot\pi/2)\,dx^n;$

and $v = x^2, \quad dv = 2\,x\,dx, \quad d^2v = 2\,dx^2, \quad d^3v = 0.$

Substituting these values in (4), we obtain

$$d^n(x^2\sin ax)/dx^n = x^2a^n\sin(ax + n\cdot\pi/2)$$
$$+ 2\,nxa^{n-1}\sin[ax + (n-1)\,\pi/2]$$
$$+ n(n-1)\,a^{n-2}\sin[ax + (n-2)\,\pi/2].$$

3. $d^n(xe^x) = e^x(x + n)\,dx^n.$

4. $d^n(x^2e^{ax})/dx^n = a^{n-2}e^{ax}[a^2x^2 + 2\,anx + n(n-1)].$

5. $d^n(x^2a^x) = a^x(\log a)^{n-2}[(x\log a + n)^2 - n]\,dx^n.$

6. $d^n(x^2\log x) = 2(-1)^{n-1}\underline{|n-3}\;x^{2-n}\,dx^n.$

82. Acceleration is the time-rate of the velocity v. Hence, if s = the distance, and a = the acceleration,

$$v = \frac{ds}{dt}, \quad a = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}. \tag{1}$$

EXAMPLES.

1. A point moves along the arc of the parabola $y^2 = 4\,px$ with the constant velocity v'; find its acceleration in the direction of each axis.

From example 20 of § 36, we obtain

$$\frac{dx}{dt} = \frac{yv'}{\sqrt{y^2 + 4\,p^2}}, \quad \frac{dy}{dt} = \frac{2\,pv'}{\sqrt{y^2 + 4\,p^2}}.$$

$$\therefore \frac{d^2x}{dt^2} = \frac{8\,p^3v'^2}{(y^2 + 4\,p^2)^2}, \quad \frac{d^2y}{dt^2} = -\frac{4\,p^2yv'^2}{(y^2 + 4\,p^2)^2}.$$

The velocities in the directions of the axes are the time-rates of x and y in the first quadrant.

Since d^2x/dt^2 is positive, the velocity in the direction of the x-axis continually increases.

Since for y positive d^2y/dt^2 is negative, the velocity in the direction of the y-axis is constantly decreasing.

2. Find the accelerations required in example 1, when the path of the point is

(1) an arc of the circle $x^2 + y^2 = r^2$,

(2) an arc of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

(3) an arc of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

3. If s denotes the number of feet a body falls in t seconds, and $g = 32.17$, $s = gt^2/2$ is the law of falling bodies in a vacuum near the earth's surface; find the velocity and the acceleration.

Ans. $v = ds/dt = gt$; $\alpha = d^2s/dt^2 = g$, a constant.

4. Given $s = ct^{1/2}$; find v and α at the end of four seconds.

CHAPTER V.

INDETERMINATE FORMS.

83. The **value of a function** of x for $x = a$ usually means the result obtained by substituting a for x in the function. When, however, this substitution gives rise to any one of the indeterminate forms

$$0/0,\ \infty/\infty,\ 0 \cdot \infty,\ \infty - \infty,\ 0^0,\ \infty^0,\ 1^\infty,$$

the definition given above is inapplicable and must be enlarged as below:

The *value of a function* for any particular value of its variable is the *limit* which the function approaches when the variable approaches this particular value as its limit.

This definition is of general application, but it is practically useful only when the ordinary and simpler definition fails.

$f(x)$ is often written without the parentheses, as fx.

The expression $fx]_a$ denotes the value of fx when $x = a$.

EXAMPLES.

By principles of limits prove that

1. $(x-a)^{1/3}/(x^2-a^2)^{1/4}]_a = 0.$

When $x = a$ this fraction assumes the indeterminate form $0/0$. Hence, to evaluate it for $x = a$ we must find its limit when $x \doteq a$.

For values of x other than a, we have

$$\frac{(x-a)^{1/3}}{(x^2-a^2)^{1/4}} \equiv \frac{(x-a)^{4/12}}{(x-a)^{3/12}(x+a)^{1/4}} \equiv \frac{(x-a)^{1/12}}{(x+a)^{1/4}}.$$

$$\therefore \underset{x \doteq a}{\text{limit}} \left[\frac{(x-a)^{1/3}}{(x^2-a^2)^{1/4}}\right] = \underset{x \doteq a}{\text{limit}} \left[\frac{(x-a)^{1/12}}{(x+a)^{1/4}}\right] = \frac{0}{\sqrt[4]{2a}} = 0.$$

That is, the given fraction equals zero when $x = a$.

2. $\left.\dfrac{x^3-a^3}{x^2-a^2}\right]_a=\dfrac{3a}{2}.$ 3. $\left.\dfrac{x^5-1}{x^3-1}\right]_1=\dfrac{5}{3}.$

4. $\left.\dfrac{(a^2-x^2)^{1/2}+(a-x)}{(a-x)^{1/2}+(a^3-x^3)^{1/2}}\right]_a=\dfrac{\sqrt{2a}}{1+a\sqrt{3}}.$

5. $(1-\cos\theta)/\sin\theta]_0=0.$

For values of θ other than zero, we have

$$\frac{1-\cos\theta}{\sin\theta}\equiv\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\equiv\tan\frac{\theta}{2}.$$

$$\therefore \underset{\theta\doteq 0}{\text{limit}}\left[\frac{1-\cos\theta}{\sin\theta}\right]=\underset{\theta\doteq 0}{\text{limit}}\left[\tan\frac{\theta}{2}\right]=0.$$

84. If $[fx/\phi x]_a=0/0$, then $[fx/\phi x]_a=[f'x/\phi'x]_a$.

That is, *if the ratio of two functions of* x *assumes the form* $0/0$ *when* x = a, *then the ratio of these functions when* x = a *is equal to the ratio of their derivatives when* x = a.

By [33] of § 55, the limit of the ratio of the increments of two variables is equal to the ratio of their differentials; hence,

$$\underset{\Delta x\doteq 0}{\text{limit}}\left[\frac{f(x+\Delta x)-fx}{\phi(x+\Delta x)-\phi x}\right]=\frac{f'x\cdot dx}{\phi'x\cdot dx}=\frac{f'x}{\phi'x}.$$

Substituting a for x and remembering that $fa=\phi a=0$, we obtain

$$\underset{\Delta x\doteq 0}{\text{limit}}\left[\frac{f(a+\Delta x)}{\phi(a+\Delta x)}\right]=\frac{f'a}{\phi'a},\ \text{or}\ \left.\frac{fx}{\phi x}\right]_a=\left.\frac{f'x}{\phi'x}\right]_a.$$

If $f'x/\phi'x]_a$ also assumes the form $0/0$, by the principle just proved we have

$$f'x/\phi'x]_a=f''x/\phi''x]_a;$$

and so on, until we obtain a fraction which does not assume the form $0/0$ when $x=a$.

EXAMPLES.

1. $\log x/(x-1)]_1 = 1.$

$$\left.\frac{\log x}{x-1}\right]_1 = \frac{0}{0}; \quad \therefore \left.\frac{\log x}{x-1}\right]_1 = \left.\frac{1/x}{1}\right]_1 = 1. \qquad \S\ 84$$

2. $(1-\cos x)/x^2]_0 = 1/2.$

$$\left.\frac{1-\cos x}{x^2}\right]_0 = \frac{0}{0}; \quad \therefore \left.\frac{1-\cos x}{x^2}\right]_0 = \left.\frac{\sin x}{2x}\right]_0 = \frac{0}{0};$$

$$\therefore \left.\frac{\sin x}{2x}\right]_0 = \left.\frac{\cos x}{2}\right]_0 = \frac{1}{2}; \quad \therefore \left.\frac{1-\cos x}{x^2}\right]_0 = \frac{1}{2}.$$

3. $\left.\dfrac{x-1}{x^n-1}\right]_1 = \dfrac{1}{n}.$

4. $\left.\dfrac{e^x-e^{-x}}{\sin x}\right]_0 = 2.$

5. $\left.\dfrac{e^x-e^{-x}-2x}{x-\sin x}\right]_0 = 2.$

6. $\left.\dfrac{a^x-b^x}{x}\right]_0 = \log\dfrac{a}{b}.$

7. $\left.\dfrac{x-\sin^{-1}x}{\sin^3 x}\right]_0 = -\dfrac{1}{6}.$

8. $\left.\dfrac{a^{x+1}-b^{x+1}}{x+1}\right]_{-1} = \log\dfrac{a}{b}.$

9. $\left.\left(\dfrac{\sin nx}{x}\right)^m\right]_0 = n^m.$

10. $\left.\dfrac{x^2-x}{1-x+\log x}\right]_1 = \infty.$

11. $\left.\dfrac{x\log(1+x)}{1-\cos x}\right]_0 = 2.$

12. $\left.\dfrac{\tan x-\sin x}{\sin^3 x}\right]_0 = \dfrac{1}{2}.$

13. $\left.\dfrac{x^x-x}{1-x+\log x}\right]_1 = -2.$

14. $\left.\dfrac{\sec^2 x-2\tan x}{1+\cos 4x}\right]_{\pi/4} = \dfrac{1}{2}.$

85. *When* $\left.\dfrac{fx}{\phi x}\right]_a$ *assumes the form* $\dfrac{\infty}{\infty}$, $\left.\dfrac{fx}{\phi x}\right]_a = \left.\dfrac{f'x}{\phi' x}\right]_a.$ (1)

When $\left.\dfrac{fx}{\phi x}\right]_a = \dfrac{\infty}{\infty}$, $\left.\dfrac{1/fx}{1/\phi x}\right]_a = \dfrac{0}{0}.$ § 74

Hence, $\underset{x \doteq a}{\text{limit}}\left[\dfrac{1/fx}{1/\phi x}\right] = \underset{x \doteq a}{\text{limit}}\left[\dfrac{f'x/(fx)^2}{\phi' x/(\phi x)^2}\right].$ § 84

If two variables have equal limits, any equimultiples of these variables have equal limits.

Hence, multiplying by $(fx)^2/(\phi x)^2$, we obtain

$$\underset{x \doteq a}{\text{limit}}\left[\frac{fx}{\phi x}\right] = \underset{x \doteq a}{\text{limit}}\left[\frac{f'x}{\phi' x}\right], \text{ or (1).}$$

EXAMPLES.

1. $\csc 2x/\csc 5x]_0 = 5/2$.

$$\frac{\csc 2x}{\csc 5x} \equiv \frac{\sin 5x}{\sin 2x} = \frac{0}{0} \text{ when } x = 0.$$

$$\therefore \left.\frac{\csc 2x}{\csc 5x}\right]_0 = \left.\frac{\sin 5x}{\sin 2x}\right]_0 = \left.\frac{5\cos 5x}{2\cos 2x}\right]_0 = \frac{5}{2}. \qquad \S\ 84$$

Here we transform the *given* fraction into an identical fraction which assumes the form $0/0$ when $x = 0$.

2. $\log x/\csc x]_0 = 0$.

$$\left.\frac{\log x}{\csc x}\right]_0 = \frac{-\infty}{\infty}; \quad \therefore \left.\frac{\log x}{\csc x}\right]_0 = \left.\frac{1/x}{-\csc x \cot x}\right]_0. \qquad \S\ 85$$

$$\frac{1/x}{-\csc x \cot x} \equiv \frac{-\sin^2 x}{x \cos x}; \quad \therefore \left.\frac{1/x}{-\csc x \cot x}\right]_0 = \left.\frac{-\sin^2 x}{x\cos x}\right]_0 = \frac{0}{0}.$$

$$\therefore \left.\frac{-\sin^2 x}{x\cos x}\right]_0 = \left.\frac{-2\sin x\cos x}{\cos x - x\sin x}\right]_0 = 0. \qquad \S\ 84$$

Here we derive a new fraction by § 85, and transform that into an identical fraction which assumes the form $0/0$ when $x = 0$.

3. $\left.\dfrac{\log x}{\cot x}\right]_0 = 0.$

4. $\left.\dfrac{\tan x}{\tan 3x}\right]_{\pi/2} = 3.$

5. $\left.\dfrac{\log(x - \pi/2)}{\tan x}\right]_{\pi/2} = 0.$

6. $\left.\dfrac{\sec x}{\sec 3x}\right]_{\pi/2} = -3.$

7. $\left.\dfrac{\log(1 - x)}{\sec(\pi x/2)}\right]_1 = 0.$

8. $\left.\dfrac{\log \tan 2x}{\log \tan x}\right]_{\pi/2} = -1.$

9. $\left.\dfrac{(e^x - 1)\tan^2 x}{x^3}\right]_0 = \left[\dfrac{e^x - 1}{x}\left(\dfrac{\tan x}{x}\right)^2\right]_0 = 1 \times 1 = 1.$

10. $\left.\dfrac{\tan x - x}{x - \sin x}\right]_0 = 2.$

11. $\left.\dfrac{1 - \sin x + \cos x}{\sin x + \cos x - 1}\right]_{\pi/2} = 1.$

86. The forms $0 \cdot \infty$ and $\infty - \infty$. A function of x which assumes the form $0 \cdot \infty$ or $\infty - \infty$ when $x = a$ is evaluated by first transforming it into an identical fraction which will assume the form $0/0$ or ∞/∞ when $x = a$.

EXAMPLES.

1. $(1-x)\tan(\pi x/2)]_1 = 2/\pi.$

Taking the reciprocal of the infinite factor, we have

$$(1-x)\tan\frac{\pi x}{2} \equiv \frac{1-x}{\cot(\pi x/2)} = \frac{0}{0} \text{ when } x = 1.$$

2. $\sec 3x \cos 7x]_{\pi/2} = 7/3.$

3. $(1-\tan x)\sec 2x]_{\pi/4} = 1.$

4. $\tan x \log \sin x]_{\pi/2} = 0.$

5. $\sin x \log x]_0 = 0.$

6. $x \log x]_0 = 0.$

7. $\left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]_1 = -\frac{1}{2}$

$$\frac{2}{x^2-1} - \frac{1}{x-1} \equiv \frac{1-x}{x^2-1} = \frac{0}{0} \text{ when } x = 1.$$

8. $\left[\frac{1}{\log x} - \frac{x}{\log x}\right]_1 = -1.$

9. $\left[\frac{2}{\sin^2 x} - \frac{1}{1-\cos x}\right]_0 = \frac{1}{2}\cdot$

10. $\left[x\tan x - \frac{\pi}{2}\sec x\right]_{\pi/2} = -1.$

11. $x\log\left(1+\frac{a}{x}\right)\Big]_x \doteq a.$

$$\underset{x=\infty}{\text{Limit}}\left[x\log\left(1+\frac{a}{x}\right)\right] = \frac{\log(1+az)}{z}\Big]_0 = a, \text{ where } z = \frac{1}{x}\cdot$$

87. The forms 0^0, ∞^0, and $1^{\pm\infty}$. When for $x = a$, a function of x assumes one of the forms 0^0, ∞^0, or $1^{\pm\infty}$, the logarithm of the function will assume the form $\pm 0\cdot\infty$, and can be evaluated by § 86. From its logarithm the value of the given function can be obtained.

EXAMPLES.

1. $x^x]_0 = 1.$

$$\log(x^x) \equiv x\log x = -0\cdot\infty \text{ when } x = 0.$$

$$x\log x]_0 = 0. \qquad \text{§ 86, example 6}$$

$$\therefore \log x^x]_0 = 0; \quad \therefore x^x]_0 = 1.$$

2. $\left(1+\frac{a}{x}\right)^x\Big]_\infty \doteq e^a; \quad \therefore \left(1+\frac{1}{x}\right)^x\Big]_\infty \doteq e.$

$$\log\left(1+\frac{a}{x}\right)^x\Big]_x = x\log\left(1+\frac{a}{x}\right)\Big]_x \doteq a. \qquad \text{§ 86, example 11}$$

$$\therefore \left(1+\frac{a}{x}\right)^x\Big]_\infty \doteq e^a; \quad \therefore \left(1+\frac{1}{x}\right)^x\Big]_\infty \doteq e.$$

3. $x^{\sin x}]_0 = 1.$

4. $(\sin x)^{\tan x}]_{\pi/2} = 1.$

5. $x^{1/(1-x)}]_1 = e^{-1}.$

6. $(\cos mx)^{n/x}]_0 = 1.$

7. $x^{e^x - 1}]_0 = 1.$

8. $(1 + ax)^{1/x}]_0 = e^a.$

9. $(\log x)^x]_0 = 1.$

10. $(e^x + x)^{1/x}]_0 = e^2.$

11. $(\cos 2x)^{1/x^2}]_0 = e^{-2}.$

12. $(\log x)^{x-1}]_1 = 1.$

88. Evaluation of derivatives of implicit functions.

When y is an implicit function of x, its derivative, though containing both x and y, is a function of x. Hence, when the derivative assumes an indeterminate form for particular values of x and y, it can be evaluated by the previous methods.

EXAMPLES.

1. Find the slope of $a^2y^2 - a^2x^2 - x^4 = 0$ at (0, 0).

Here $$\frac{dy}{dx} = \frac{2a^2x + 4x^3}{2a^2y} = \frac{0}{0}, \text{ when } x = y = 0.$$

Hence, $$\left.\frac{dy}{dx}\right]_{0,0} = \left.\frac{2a^2x + 4x^3}{2a^2y}\right]_{0,0} = \left.\frac{2a^2 + 12x^2}{2a^2 \cdot dy/dx}\right]_{0,0} = \left.\frac{1}{dy/dx}\right]_{0,0};$$

$\therefore (dy/dx)^2]_{0,0} = 1$, or $dy/dx]_{0,0} = \pm 1$.

2. Find the slope of $y^3 = ax^2 - x^3$ at (0, 0).

Here $$\left.\frac{dy}{dx}\right]_{0,0} = \left.\frac{2ax - 3x^2}{3y^2}\right]_{0,0} = \left.\frac{2a - 6x}{6y \cdot dy/dx}\right]_{0,0};$$

$$\therefore \left.\left(\frac{dy}{dx}\right)^2\right]_{0,0} = \left.\frac{2a - 6x}{6y}\right]_{0,0} = \frac{2a}{0} = \infty, \text{ or } \left.\frac{dy}{dx}\right]_{0,0} = \pm\infty.$$

3. Find the slope of $x^3 - 3axy + y^3 = 0$ at (0, 0).

Ans. $dy/dx]_{0,0} = 0$ or ∞.

4. Find the slope of $x^4 - a^2xy + b^2y^2 = 0$ at (0, 0).

Ans. $dy/dx]_{0,0} = 0$ or a^2/b^2.

5. Find the slope of $(y^2 + x^2)^2 - 6axy^2 - 2ax^3 + a^2x^2 = 0$ at (0, 0) and $(a, 0)$.

Ans. $dy/dx]_{0,0} = \pm\infty$; $dy/dx]_{a,0} = \pm 1/2$.

CHAPTER VI.

EXPANSION OF FUNCTIONS.

89. A **series** is a succession of terms whose values are all determined by any one law. A series is *finite* or *infinite* according as the number of its terms is limited or unlimited.

The *sum* of a finite series is the sum of all its terms.

The *sum* of an *infinite* series is the *limit* of the sum of its first n terms as n increases indefinitely. When such a limit exists, the series is said to be *convergent;* when no such limit exists, the series is *divergent* and has no sum.

For example, the series

$$1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^{n-1} + \cdots$$

is convergent; for the sum of its first n terms $\doteq 2$ when $n = \infty$.

The series

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots$$

is divergent; for the sum of its first n terms does not approach a limit when $n = \infty$.

90. To **expand** a function is to find a series the sum of which shall equal the function. Hence, the *expansion* of a function is either a *finite* or a *convergent infinite* series.

When the expansion is an infinite series, the difference between the function and the sum of the first n terms of the series is called the **remainder after n terms.** When $n = \infty$, this remainder must evidently approach zero as its limit.

For example, by division we obtain

$$\frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \frac{x^n}{1-x}. \qquad (1)$$

Here $x^n/(1-x)$ is the *remainder after n terms.* When $n=\infty$ and $x>-1$ and <1, this remainder $\doteq 0$, and therefore the sum of n terms of the series $\doteq$ the function; but when $n=\infty$ and $x>1$ or <-1, the remainder increases arithmetically, and therefore the sum of n terms of the series *diverges* more and more from the value of the function.

Hence, the series in (1) is convergent and its sum equals the function only for values of x between -1 and $+1$.

Some functions may be expanded by division, as above; some by indeterminate coefficients; others by the binomial theorem; and so on. The binomial theorem, the logarithmic series, the exponential series, etc., are all particular cases of *Taylor's theorem,* which is stated and proved in § 92.

For the proof of this theorem we need the following lemma:

91. Lemma. *If ϕz and $\phi' z$ are each continuous between* a *and* a + h, *and $\phi a \equiv \phi(a+h) \equiv 0$; $\phi' z$ must equal zero for at least one value of z between* a *and* a + h; *that is, $\phi'(a+\theta h)=0$, where θ is some positive proper fraction.*

For if ϕz is continuous and $\phi a \equiv \phi(a+h) \equiv 0$; then, as z changes from a to $a+h$, ϕz must first increase and then decrease, or first decrease and then increase; hence, $\phi' z$ must change from + to − or from − to +; and therefore, if continuous, it must pass through 0 for some value of z between a and $a+h$. Denoting this value of z by $a+\theta h$ where θ has some value between 0 and + 1, we have $\phi'(a+\theta h)=0$.

92. Taylor's theorem. *When $fz, f'z, f''z, \cdots, f^n z$ are each continuous between x and $x+h$,*

$$f(x+h) \equiv$$
$$fx + f'x\frac{h}{1} + f''x\frac{h^2}{\underline{|2}} + \cdots + f^{n-1}x\,\frac{h^{n-1}}{\underline{|n-1}} + f^n(x+\theta h)\frac{h^n}{\underline{|n}}, \quad \text{(A)}$$

where the last term is the remainder after n terms, and θ is some positive proper fraction.

Let $Ph^n/\underline{|n}$ denote the remainder after n terms when $x = a$; then we have

$$f(a+h) \equiv$$
$$fa + f'a\frac{h}{1} + f''a\frac{h^2}{\underline{|2}} + \cdots + f^{n-1}a\frac{h^{n-1}}{\underline{|n-1}} + P\frac{h^n}{\underline{|n}}. \qquad (1)$$

We proceed to find the value of P.

Putting $h = b - a$ in (1), and transposing, we obtain

$$fb - fa - f'a\frac{b-a}{1} - f''a\frac{(b-a)^2}{\underline{|2}} - f'''a\frac{(b-a)^3}{\underline{|3}} - \cdots$$
$$- f^{n-1}a\frac{(b-a)^{n-1}}{\underline{|n-1}} - P\frac{(b-a)^n}{\underline{|n}} = 0. \qquad (2)$$

Let ϕz represent the function of z obtained by substituting z for a in the first member of (2); then

$$\phi z \equiv fb - fz - f'z\frac{b-z}{1} - f''z\frac{(b-z)^2}{\underline{|2}} - \cdots$$
$$- f^{n-1}z\frac{(b-z)^{n-1}}{\underline{|n-1}} - P\frac{(b-z)^n}{\underline{|n}}. \qquad (3)$$

Differentiating (3) to obtain $\phi' z$, we find that the terms of the second member destroy each other in pairs with the exception of the last two, and obtain

$$\phi' z \equiv -\frac{(b-z)^{n-1}}{\underline{|n-1}}f^n z + \frac{(b-z)^{n-1}}{\underline{|n-1}}P. \qquad (4)$$

By hypothesis fz, $f'z$, $\cdots$, $f^n z$ are continuous between a and $a + h$; hence, from (3) and (4), it follows that ϕz and $\phi' z$ are continuous between a and $a + h$.

Putting a for z in (3), by (2) we have $\phi a \equiv 0$.

Putting b for z in (3), we have $\phi b \equiv 0$, *i.e.* $\phi(a+h) \equiv 0$.

Hence, by § 91 we have

$$\phi'(a + \theta h) = 0, \qquad (5)$$

where θ denotes some positive proper fraction.

Putting $a + \theta h$ for z in (4), by (5) we obtain

$$P = f^n(a + \theta h).$$

Substituting this value of P in (1), and then putting x for a, since a is any value of x, we obtain (A).

Formula (A), called Taylor's theorem, was first published by Brook Taylor in 1715.

Cor. 1. Putting $x = 0$ in (A) and then substituting x for h, we obtain

$$fx = f0 + f'0\,\frac{x}{1} + f''0\,\frac{x^2}{\underline{|2}} + \cdots + f^{n-1}0\,\frac{x^{n-1}}{\underline{|n-1}} + f^n(\theta x)\,\frac{x^n}{\underline{|n}}. \quad \text{(B)}$$

Formula (B), called Stirling's or Maclaurin's theorem, was first given by James Stirling in 1717. It is a special case of Taylor's theorem.

Cor. 2.* Denoting the remainder after n terms in (A) by R_T, and in (B) by R_M, we have

$$R_T = f^n(x + \theta h)\,\frac{h^n}{\underline{|n}}, \quad R_M = f^n(\theta x)\,\frac{x^n}{\underline{|n}}. \qquad (6)$$

The form of R_T in (6) is due to Lagrange.

Cor. 3. If $R_T \doteq 0$ when $n = \infty$, the series in (A) is the expansion of $f(x + h)$. If $R_M \doteq 0$ when $n = \infty$, the series in (B) is the expansion of fx.

Cor. 4. If $f^n(x)$ increases (or decreases) from $f^n(x)$ to $f^n(x + h)$, and we take the sum of the first n terms in (A) as the value of $f(x + h)$, the *error* lies between

$$f^n(x) \cdot h^n / \underline{|n} \text{ and } f^n(x + h)\, h^n / \underline{|n}.$$

If $f^n(x)$ increases (or decreases) from $f^n(0)$ to $f^n(x)$, and we take the sum of the first n terms in (B) as the value of $f(x)$, the error lies between

$$f^n(0)\, x^n / \underline{|n} \text{ and } f^n(x)\, x^n / \underline{|n}.$$

---

* Cors. 2, 3, 4, §§ 93, 96, and the proofs for convergency in §§ 94–98 may be omitted in the first reading of this chapter.

93. Since $\frac{x^n}{\underline{|n}} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{n-2} \cdot \frac{x}{n-1} \cdot \frac{x}{n}$,

$$x^n / \underline{|n} \doteq 0 \text{ when } x \text{ is finite and } n = \infty.$$

Cor. When $n = \infty$ and x is finite,

$$R_T \doteq 0 \text{ when } f^n(x + \theta h) \text{ is finite,}$$

and $$R_M \doteq 0 \text{ when } f^n(\theta x) \text{ is finite.}$$

94. *To expand sin* x *and cos* x.

Sin x is a particular case of fx; hence, to expand sin x we use Maclaurin's theorem, or (B).

Here
$$fx = \sin x, \qquad \therefore f0 = 0;$$
$$f'x = \cos x, \qquad \therefore f'0 = 1;$$
$$f''x = -\sin x, \qquad \therefore f''0 = 0;$$
$$f'''x = -\cos x, \qquad \therefore f'''0 = -1.$$

Since $$f^{\text{IV}}x = \sin x = fx,$$

the four values given above will recur in sets.

Substituting these values in (B), we obtain

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{\underline{|2n-1}} + \cdots. \quad (1)$$

The nth term in (1) is readily written out by inspection.

Proof of (1).* Since $f^n x = \sin(x + n \cdot \pi/2)$, fx and all its successive derivatives are continuous and finite for all values of x. Hence, by Cor. of § 93 and Cor. 3 of § 92, the series in (1) is the expansion of sin x for all finite values of x.

---

* For a discussion of convergent series consult Taylor's "College Algebra," Osgood's "Introduction to Infinite Series," or some more extended work on the Calculus. For the proof of convergency the scope of this work limits us to the use of the remainder after n terms in (A) or (B).

In like manner we obtain and prove

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots + \frac{(-x^2)^{n-1}}{\underline{|2(n-1)}} + \cdots. \qquad (2)$$

Identity (2) could be obtained by differentiating (1).

95. *To deduce the exponential series.*

$$a^x = 1 + \frac{x \log a}{1} + \frac{(x \log a)^2}{\underline{|2}} + \cdots + \frac{(x \log a)^{n-1}}{\underline{|n-1}} + \cdots. \qquad (1)$$

Proof of (1). Here $f^n x = (\log a)^n a^x$.

Hence, when a is positive, fx and all its successive derivatives are *continuous* for all values of x.

$$R_M = f^n(\theta x)\frac{x^n}{\underline{|n}} = \frac{(x \log a)^n}{\underline{|n}} a^{\theta x}.$$

When x is finite, $a^{\theta x}$ is finite.

By § 93, $(x \log a)^n / \underline{|n} \doteq 0$ when $n = \infty$ and $x \log a$ is finite.

Hence, $R_M \doteq 0$ when $n = \infty$ and x is finite.

Therefore, the series in (1) is the expansion of a^x when a is positive and x is finite.

Cor. 1. **Value of e^x.** Putting $a = e$ in (1), we obtain

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots + \frac{x^{n-1}}{\underline{|n-1}} + \cdots. \qquad (2)$$

Cor. 2. **Value of e.** Putting $x = 1$ in (2), we obtain

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots + \frac{1}{\underline{|n-1}} + \cdots. \qquad (3)$$

$$= 2.718281 \cdots.$$

96. Second form of remainder. If we denote the remainder after n terms in (1) of § 92 by $P_1 h$, and proceed as before, we obtain

$$R_T = f^n(x + \theta h)\frac{(1-\theta)^{n-1} h^n}{\underline{|n-1}} \cdot R_M = f^n(\theta x)\frac{(1-\theta)^{n-1} x^n}{\underline{|n-1}}.$$

97. *To deduce the logarithmic series.*

$$\log_a(1+x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots\right). \quad (1)$$

Proof of (1). $f^n x = (-1)^{n-1}\underline{|n-1}/(1+x)^n$.

When $x < -1$, $\log(1+x)$ has no real value.

When $x = -1$, the odd derivatives are discontinuous.

When $x > -1$, fx and all its successive derivatives are continuous.

Using the second form of R_M, we obtain

$$R_M = \left(\frac{x - \theta x}{1 + \theta x}\right)^n \cdot \frac{(-1)^{n-1}}{1-\theta}.$$

The second factor in R_M is finite, and the first factor $\doteq 0$ when $n = \infty$ and $x > -1$ and < 1 or $x = 1$.

Therefore, the series in (1) is the expansion of $\log(1+x)$ when $x > -1$ and $< +1$ or $x = 1$.

Putting $x = 1$ in (1), we obtain $\log_a 2$.

Putting $x = 1/2$, we obtain $\log_a(3/2)$, or $\log_a 3 - \log_a 2$; etc.

Cor. 1. To deduce a series more rapidly convergent than that in (1), we put $-x$ for x in (1) and obtain

$$\log_a(1-x) = m\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right). \quad (2)$$

Subtracting (2) from (1), we obtain

$$\log_a \frac{1+x}{1-x} = 2m\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right). \quad (3)$$

Let $$x = \frac{1}{2z+1}; \text{ then } \frac{1+x}{1-x} = \frac{z+1}{z}. \quad (4)$$

Substituting in (3) the values in (4), we obtain

$$\log_a \frac{z+1}{z} = 2m\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right). \quad (5)$$

$$\therefore \log_a(z+1) = \log_a z + 2\,m\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right). \quad (6)$$

When $z > 0$, $0 < x < 1$; hence, the series in (6) is convergent for all positive values of z.

$\text{Log}_a(z+1)$ can be readily computed when $\log_a z$ is known.

Cor. 2. If $m = 1$, $a = e$ and (5) becomes

$$\log\frac{z+1}{z} = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right). \qquad (7)$$

Dividing (5) by (7) and denoting $(z+1)/z$ by N, we have

$$\log_a N/\log N = m, \text{ or } \mathbf{\log_a N = m \log N}. \qquad (8)$$

Cor. 3. **Value of m.** Putting $N = a$ in (8), we obtain

$$1 = m\log a, \text{ or } \mathbf{m = 1/\log a}. \qquad (9)$$

Cor. 4. **Value of M.** If M denotes the modulus of the common system whose base is 10, from (9) we have

$$M = \frac{1}{\log 10} = \frac{1}{2.302585} = 0.434294\cdots.$$

98. *To deduce the binomial theorem, or*

$$(x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{\lfloor 2}\cdot x^{m-2}h^2$$
$$+ \frac{m(m-1)(m-2)}{\lfloor 3}x^{m-3}h^3 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+2)}{\lfloor n-1}x^{m-n+1}h^{n-1} + \cdots. \qquad (1)$$

$(x+h)^m$ is a particular case of $f(x+h)$; hence, to expand $(x+h)^m$ we use Taylor's theorem, or (A).

$$\text{Here}\quad f(x+h) = (x+h)^m; \qquad \therefore fx = x^m,$$

$$f'x = mx^{m-1}, \qquad f''x = m(m-1)x^{m-2},$$

$$f'''x = m(m-1)(m-2)x^{m-3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^{n-1}x = m(m-1)\cdots(m-n+2)x^{m-n+1}.$$

Substituting these values in (A), we obtain (1).

Proof of (1). $f^n x = m(m-1)\cdots(m-n+1)x^{m-n}$.

Hence, fx and all its successive derivatives are continuous for all values of x.

Using the second form of R_T, we obtain

$$R_T = \frac{m(m-1)\cdots(m-n+1)}{\underline{|n-1}} \cdot \left(\frac{h-\theta h}{x+\theta h}\right)^n \cdot \frac{(x+\theta h)^m}{1-\theta}.$$

When $x > h$ arithmetically, and $n = \infty$, the product of the first and second factors $\doteq 0$; hence, $R_T \doteq 0$.

Hence, the binomial theorem holds true when the first term is greater than the second arithmetically.

When m is a positive integer, we obtain

$$f^{m+1}x \equiv 0, \ f^{m+2}x \equiv 0, \ \cdots;$$

hence, in this case, the expansion in (1) is a finite series of $m+1$ terms.

99. Failure of Maclaurin's theorem. The successive derivatives of $\log x$ are $\alpha\varphi$ and discontinuous when $x = 0$; hence, (B) fails to expand $\log x$.

For a like reason, (B) fails to expand

$$\cot x, \quad \csc x, \quad \text{vers}^{-1} x, \quad a^{1/x}, \quad \sin(1/x), \cdots.$$

When (B) fails to expand fx, (A) will fail to expand $f(x+h)$ for $x = 0$.

For example, when $x = 0$, (A) fails to expand $\log(x+h)$, $\cot(x+h)$, $\text{vers}^{-1}(x+h)$, $\cdots$.

(A) may fail to expand $f(x+h)$ for other values than $x = 0$.

The limits between which any expansion holds true should be carefully determined.

EXAMPLES.

1. $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots.$$

2. $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$$

Here $f'x = (1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots;$

$$\therefore f''x = -2x + 4x^3 - 6x^5 + \cdots, \text{ etc.}$$

3. $$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots.$$

4. From the series in example 3 find the value of π.

Putting $x = 1/2$, we obtain

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6} = \frac{1}{2}\left(1 + \frac{1}{24} + \frac{3}{640} + \frac{5}{7168} + \cdots\right);$$

$$\therefore \pi = 3.141592\cdots.$$

5. $$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots.$$

6. $$e^{\sin x} = 1 + x + \frac{x^2}{\lfloor 2} - \frac{3x^4}{\lfloor 4} + \frac{8x^5}{\lfloor 5} - \frac{3x^6}{\lfloor 6} + \cdots.$$

7. $$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots.$$

8. $$\begin{aligned}\sin(x + h) &= \sin x\left(1 - \frac{h^2}{\lfloor 2} + \frac{h^4}{\lfloor 4} - \frac{h^6}{\lfloor 6} + \cdots\right)\\ &\quad + \cos x\left(h - \frac{h^3}{\lfloor 3} + \frac{h^5}{\lfloor 5} - \frac{h^7}{\lfloor 7} + \cdots\right)\\ &= \sin x \cos h + \cos x \sin h. \qquad \S\ 94\end{aligned}$$

9. $$\cos(x + h) = \cos x \cos h - \sin x \sin h.$$

10. $\log_a(x + h)$

$$= \log_a x + m\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots + \frac{(-1)^{n-1}h^n}{nx^n} + \cdots\right).$$

11. $a^{x+h} = a^x\left(1 + \frac{h \log a}{1} + \frac{(h \log a)^2}{\lfloor 2} + \cdots + \frac{(h \log a)^{n-1}}{\lfloor n-1} + \cdots\right).$

12. $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2^3} - \frac{x^4}{2^3 \lfloor 4} + \cdots.$

13. If $fx = f(-x)$, the expansion of fx will contain only even powers of x; while if $fx = -f(-x)$, the expansion of fx will involve only odd powers of x.

14. What powers of x will appear in the expansion of $\sin x$? $\cos x$? $\tan x$? $\sec x$? $\sin^{-1} x$? $\tan^{-1} x$? $(e^x + 1)/(e^x - 1)$?

15. Regarding h as an increment of x, find from (A) the value of the corresponding increment of fx, h being reckoned (1) from any value of x, (2) from $x = 0$.

Compare the second result with (B).

16. Prove geometrically that

$$f(x + h) = fx + hf'(x + \theta h),$$

fx and $f'x$ being continuous between x and $x + h$.

17. When $x = i$, prove that

$$x - \sin x = v_3 i^3; \qquad 1 - \cos x = v_2 i^2; \qquad \S\ 94$$
$$a^x - 1 = v_1 i; \qquad x - \tan x = v_3 i^3; \qquad \S\ 95$$
$$x - \tan^{-1} x = v_3 i^3; \qquad 1 - \sec x = v_2 i^2.$$

18. From (2) of § 95 and (1) and (2) of § 94, show that

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1} \sin x,$$

and

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1} \sin x.$$

19. From the results in example 18, obtain the *exponential values*

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}, \quad \cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

20. Find the limits of the error when we take the sum of n terms of the series as the value of a^x; $\log_a(1 + x)$; $(x + h)^m$.

CHAPTER VII.

MAXIMA AND MINIMA.

100. A **maximum** of fx is a value of fx which is *greater* than those immediately preceding and immediately following. A **minimum** of fx is a value which is *less* than those immediately preceding and following. In defining and discussing maxima and minima of fx, it is assumed that x *increases continuously*, and that fx is a *continuous one-valued* function.

101. f'x *is positive immediately before and negative immediately after a maximum of* fx ; *also* f'x *is negative immediately before and positive immediately after a minimum of* fx.

For fx is an *increasing* function immediately *before* and a *decreasing* function immediately *after* a maximum ; also, fx is a decreasing function immediately before and an increasing function immediately after a minimum (§ 12, Cor. 1).

102. *Any value of* x *which renders* fx *a maximum or a minimum is a root of* $f'x = 0$ *or* $f'x = \infty$.

From § 101 it follows that $f'x$ changes its quality when fx passes through either a maximum or a minimum.

When $f'x$ is continuous for all values of x, $f'x$ must pass through zero to change its quality.

When $f'x$ is a fraction whose denominator becomes zero for some finite value of x, $f'x$ may change its quality by becoming ∞.

For example, when $x - 2 = 0$ or $x = 2$, the fraction $a / (x - 2)$ becomes ∞ and changes its quality. Again, $\tan x$, or $\sin x / \cos x$, becomes ∞ and changes its quality when $\cos x = 0$, or $x = \pi / 2$.

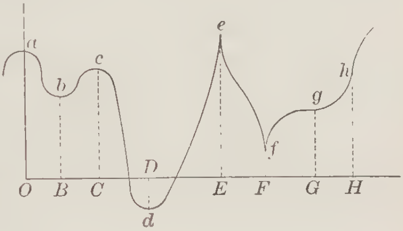
The converse of this theorem is not true; that is, any root of $f'x = 0$ or $f'x = \infty$ does not necessarily render fx either a maximum or a minimum. These roots are simply the *critical* values of x, for each of which the function is to be examined.

103. Geometric illustration.

Let *adefh* be the locus of $y = fx$.

Then fx will be represented by the ordinate of the point (x, y), and $f'x$ by the slope of the locus at the point (x, y).

By definition, the ordinates Oa, Cc, and Ee represent maxima of fx; while Bb, Dd, and Ff represent minima (§ 100).



The slope $f'x$ is positive immediately before a maximum ordinate, and negative immediately after; while the slope is negative immediately before a minimum ordinate and positive after (§ 101).

The slope $f'x$ is 0 or ∞ at any point whose ordinate fx is either a maximum or a minimum. The slope $f'x$ is *discontinuous* at the points e and f, where it changes its quality by becoming ∞ (§ 102).

The slope $f'x$ is 0 at g, and ∞ at h; but it does not change its quality at either point, and neither Gg nor Hh is a maximum or a minimum ordinate.

NOTE. If the points e and f were shooting points (§ 153) instead of cusps (§ 151), $f'x$ would change abruptly from a positive finite value to a negative value, or *vice versa;* hence, § 102 is not strictly true when the locus of $y = f(x)$ has a shooting point.

104. Maxima and minima occur alternately. For between two maxima a function must change from a decreasing to an increasing function, and hence pass through a minimum. For a like reason, between two minima a function passes through a maximum.

This principle is evident also from the locus in § 103.

105. *Let* a *be a critical value obtained from* **f'x = o,** *and let* a *be substituted for* x *in the successive derivatives of* fx.

If the first derivative which does not vanish is of an even order, fa *is a maximum or a minimum of* fx *according as this derivative is negative or positive.*

If the first derivative which does not vanish is of an odd order, fa *is neither a maximum nor a minimum of* fx.

Since $f'a = 0$, from Taylor's theorem we have

$$f(a-h) - fa = f''a \cdot h^2 / 2 - f'''a \cdot h^3 / \lfloor\underline{3} + f^{\text{IV}}a \cdot h^4 / \lfloor\underline{4} - \cdots, \quad (1)$$

$$f(a+h) - fa = f''a \cdot h^2 / 2 + f'''a \cdot h^3 / \lfloor\underline{3} + f^{\text{IV}}a \cdot h^4 / \lfloor\underline{4} + \cdots. \quad (2)$$

If h be taken very small, the quality of the second member of (1) or (2) will be that of its first term ; hence,

If $f''a$ is $-$, $fa > f(a-h)$ and $fa > f(a+h)$;

that is, fa is a maximum of fx. § 100

If $f''a$ is $+$, $fa < f(a-h)$ and $fa < f(a+h)$;

that is, fa is a minimum of fx. § 100

If $f''a = 0$ and $f'''a$ is not zero, fa is evidently neither a maximum nor a minimum of fx.

If $f''a \equiv f'''a \equiv 0$, fa will evidently be a maximum or a minimum according as $f^{\text{IV}}a$ is $-$ or $+$; and so on.

Ex. Examine $4x^3 - 15x^2 + 12x - 1$ for maxima and minima.

Here $f'x \equiv 12x^2 - 30x + 12$,

and $f''x \equiv 24x - 30$.

The roots of $f'x \equiv 12x^2 - 30x + 12 = 0$ are 2 and $1/2$; hence, the only *critical* values of x are 2 and $1/2$.

$$f''(2) \equiv +18; \quad \therefore f(2), \text{ or } -5, \text{ is a minimum of } fx;$$

$$f''(1/2) \equiv -18; \quad \therefore f(1/2), \text{ or } 7/4, \text{ is a maximum of } fx.$$

Note. The student should illustrate these properties of this function by constructing the locus of $y = 4x^3 - 15x^2 + 12x - 1$.

106. *Let* a *be a critical value given by either* $f'x = 0$ *or* $f'x = \infty$, *and let* h *be a very small positive number; then*

If $f'(a - h)$ *is positive and* $f'(a + h)$ *is negative,*

fa *is a maximum of* fx. § 101

If $f'(a - h)$ *is negative and* $f'(a + h)$ *is positive,*

fa *is a minimum of* fx. § 101

If $f'(a - h)$ *and* $f'(a + h)$ *have the same quality,*

fa *is neither a maximum nor a minimum of* fx.

107. Auxiliary principles. By the following obvious principles we may often simplify the solution of problems in maxima and minima:

(i) Since fx and $\log (fx)$ increase and decrease together, any value of x which renders fx a maximum or a minimum renders $\log (fx)$ a maximum or a minimum; and conversely.

(ii) Since when fx increases its reciprocal decreases, any value of x which renders fx a maximum or a minimum renders its reciprocal a minimum or a maximum.

(iii) Any value of x which renders $c\,(fx)$, c being positive, a maximum or a minimum renders fx a maximum or a minimum; and conversely. If c is negative and fa is a maximum, $c\,(fa)$ is a minimum.

(iv) Any value of x which renders $c + fx$ a maximum or a minimum renders fx a maximum or a minimum; and conversely.

(v) Any value of x which renders fx positive, and a maximum or a minimum, renders $(fx)^n$ a maximum or a minimum, n being any positive whole number.

EXAMPLES.

Examine fx for maxima and minima when

1. $fx \equiv x^3 - 9x^2 + 15x - 3$.

$f''(1)$ is $-$; $\therefore f(1)$, or 4, is a maximum of fx.

$f''(5)$ is $+$; $\therefore f(5)$, or -28, is a minimum of fx.

2. $fx \equiv x^5 - 5x^4 + 5x^3 - 1$.

Ans. $f(1)$, or 0, is a max.; $f(3)$, or -28, is a min.

3. $fx \equiv 3x^5 - 125x^3 + 2160x$.

Ans. $f(-4)$ and $f(3)$ are max.; $f(-3)$ and $f(4)$ are min.

4. $fx \equiv x^3 - 3x^2 + 3x + 7$.

Here $f'(1) \equiv 0$, $f''(1) \equiv 0$, and $f'''(1) \equiv 6$; hence, $f(1)$ is neither a maximum nor a minimum of fx (§ 105).

5. $fx \equiv 2x^3 - 21x^2 + 36x - 20$.

6. $fx \equiv x^3 - 3x^2 + 6x + 7$.

Ans. No *real* value of x renders $f(x)$ a max. or a min.

7. Examine $c + \sqrt{4a^2x^2 - 2ax^3}$ for maxima and minima.

By § 107 any value of x which renders $c + \sqrt{4a^2x^2 - 2ax^3}$ a maximum or a minimum renders

$$\sqrt{4a^2x^2 - 2ax^3}, \quad 4a^2x^2 - 2ax^3, \quad \text{or} \quad 2ax^2 - x^3$$

a maximum or a minimum.

Hence, we let $fx \equiv 2ax^2 - x^3$, etc.

Ans. c is a min.; and $c + 8a^2\sqrt{3}/9$ is a max.

8. Examine $b + c(x - a)^{2/3}$ for maxima and minima.

Let $fx \equiv (x - a)^2$. *Ans.* b is a min.

9. Examine $(x - 1)^4(x + 2)^3$ for maxima and minima.

$$f'x = (x - 1)^3(x + 2)^2(7x + 5).$$

Hence, the critical values are 1, -2, and $-5/7$.

$f'(1 - h)$ is $-$, and $f'(1 + h)$ is $+$;

$\therefore f(1)$, or 0, is a min. of fx. § 106

$f'(-5/7 - h)$ is $+$, and $f'(-5/7 + h)$ is $-$;

$\therefore f(-5/7)$ is a max. of fx.

$f'(-2 - h)$ and $f'(-2 + h)$ are both $+$;

hence, $f(-2)$ is neither a maximum nor a minimum of fx.

In this example the method in § 106 is preferable to that in § 105.

10. Examine $\dfrac{(a-x)^3}{a-2x}$ for maxima and minima.

$f'x \equiv (a-x)^2(4x-a)/(a-2x)^2.$

$f'x = 0$ gives $x = a$ or $a/4$;

$f'x = a\varphi$ gives $(a-2x)^2 = 0$, or $x = a/2$. § 102

$f'(a/4-h)$ is $-$, and $f'(a/4+h)$ is $+$;

$\therefore f(a/4)$ is a min. of fx.

When $x = a$, or $a/2$, $f'x$ does not change its quality; hence, neither fa nor $f(a/2)$ is a maximum or a minimum of fx.

11. Examine $\dfrac{x^2-7x+6}{x-10}$ for maxima and minima.

Ans. $f(4)$ is a max.; $f(16)$ a min.

12. Examine $\dfrac{(x+2)^3}{(x-3)^2}$ for maxima and minima.

Ans. $f(3)$ is a max; $f(13)$ a min.

13. If $fx \equiv x(x+a)^2(a-x)^3$, $f(-a)$ and $f(a/3)$ are maxima, and $f(-a/2)$ is a minimum.

14. $\sqrt{2}$ is a maximum of $\sin\theta + \cos\theta$.

15. e is a minimum of $x/\log x$.

16. 1 is a maximum of $2\tan\theta - \tan^2\theta$.

17. $e^{1/e}$ is a maximum of $x^{1/x}$.

18. $3\sqrt{3}/4$ is a maximum of $\sin\theta(1+\cos\theta)$.

19. $\theta/(1+\theta\tan\theta)$ is a maximum when $\theta = \cos\theta$.

Examine the reciprocal of this function for maxima and minima.

20. Show that 2 is a maximum ordinate and -26 a minimum ordinate of the curve $y = x^5 - 5x^4 + 5x^3 + 1$.

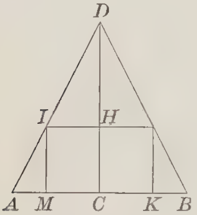
21. Show that $a/3$ is a maximum ordinate and 0 a minimum ordinate of the curve $y = (2x-a)^{1/3}(x-a)^{2/3}$.

22. Show that $3\sqrt{3}/16$ is a maximum ordinate of the curve

$$y = \sin^3 x \cos x.$$

PROBLEMS IN MAXIMA AND MINIMA.

1. Find the altitude of the maximum cylinder that can be inscribed in a given right cone.



Let DAB be a section through the axis of the cone and the inscribed cylinder.

Let $a = DC$, $b = AC$, $y = MC$, $x = IM$, and $V =$ the volume of the cylinder;

then $$V = \pi x y^2.$$

From the similar triangles ADC and IDH,

$$y = (b/a)(a - x);$$

$$\therefore V = \pi (b/a)^2 x (a - x)^2.$$

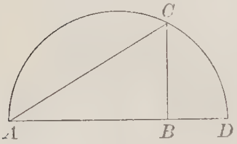
V will be a maximum when $x(a - x)^2$ is a maximum.

Hence, let $$fx \equiv x(a - x)^2, \text{ etc.}$$

Ans. The altitude of the cylinder $= 1/3$ that of the cone.

2. Find the altitude of the maximum cone that can be inscribed in a sphere whose radius is r.

Let ACD and ACB be the semicircle and the triangle which generate the sphere and the cone, respectively.



Let $x = AB$, $y = BC$, and $V =$ the volume of the cone;

then $$V = \tfrac{1}{3} \pi x y^2.$$

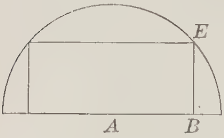
$$y^2 = AB \cdot BD = x(2r - x);$$

$$\therefore V = \tfrac{1}{3} \pi x^2 (2r - x).$$

V will be a maximum when $x^2(2r - x)$ is a maximum.

Ans. The altitude of the cone $= 4/3$ the radius of the sphere.

3. Find the altitude of the maximum cylinder that can be inscribed in a sphere whose radius is r.



Let $x = AB$, and $y = BE$;

then $$V = 2\pi x y^2 = 2\pi x (r^2 - x^2).$$

Ans. Altitude $= 2r\sqrt{3}/3$.

4. The capacity of a closed cylindrical vessel is c; find the ratio of its altitude to the diameter of its base when its entire inner surface is a minimum; find its altitude.

Let u equal the radius of the base, x the altitude, and S the entire inner surface; then

$$c = \pi x u^2, \tag{1}$$

and

$$S = 2\pi u^2 + 2\pi x u. \tag{2}$$

From (1), $$du/dx = -u/2x. \tag{3}$$

From (2), $$dS = 4\pi u\,du + 2\pi x\,du + 2\pi u\,dx. \tag{4}$$

When S is a minimum, $dS/dx = 0$, or $dS = 0$; hence,

$$du/dx = -u/(2u + x). \tag{5}$$

From (3) and (5), we obtain

$$2x = 2u + x, \text{ or } x = 2u. \tag{6}$$

Hence, as S evidently has a minimum value, it is a minimum when the altitude of the cylinder is equal to the diameter of its base.

From (1) and (6), we find the altitude

$$x = 2\sqrt[3]{c/2\pi}.$$

5. Find the maximum rectangle which can be inscribed in the ellipse

$$x^2/a^2 + y^2/b^2 = 1. \tag{1}$$

Let (x, y) be the vertex of the rectangle in the first quadrant, and let u denote the area; then

$$u = 4xy. \tag{2}$$

Differentiating (1) and (2), and proceeding as in example 4, we find that the maximum area is $2ab$.

6. Find the maximum cylinder which can be inscribed in an oblate spheroid whose semi-axes are a and b.

The ellipse which generates the spheroid is

$$x^2/a^2 + y^2/b^2 = 1. \tag{1}$$

Let (x, y) be the vertex in the first quadrant of the rectangle which generates the inscribed cylinder; then

$$V = 2\pi y x^2. \tag{2}$$

7. The capacity of a cylindrical vessel with open top being constant, what is the ratio of its altitude to the radius of its base when its inner surface is a minimum?

8. A square piece of sheet lead has a square cut out at each corner; find the side of the square cut out when the remainder of the sheet will form a vessel of maximum capacity.

9. The radius of a circular piece of paper is r; find the arc of the sector which must be cut from it that the remaining sector may form the convex surface of a cone of maximum volume.

Ans. Arc $= 2\pi r(1 - \sqrt{6}/3)$.

Let $x =$ the altitude of the cone;

then $V = \pi x(r^2 - x^2)/3$.

10. A person, being in a boat 3 miles from the nearest point of the beach, wishes to reach in the shortest time a place 5 miles from that point along the shore; supposing he can walk 5 miles an hour, but row only at the rate of 4 miles an hour, required the place where he must land.

Ans. 1 mile from the place to be reached.

11. Find the maximum right cone that can be inscribed in a given right cone, the vertex of the required cone being at the centre of the base of the given cone.

Ans. The ratio of their altitudes is 1 : 3.

12. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and the breadth of the window when the quantity of light admitted is a maximum.

Ans. The radius of the semicircle = the height of the rectangle.

13. Prove that, of all circular sectors having the same perimeter c, the sector of maximum area is that in which the circular arc is double the radius.

Let $x =$ the radius of the sector;

then $\text{area} = \dfrac{x(c - 2x)}{2}$.

14. Find the maximum convex surface of a cylinder inscribed in a cone whose altitude is b, and the radius of whose base is a.

Ans. Maximum surface $= \pi ab/2$.

15. Find the altitude of the cylinder of maximum convex surface that can be inscribed in a given sphere whose radius is r.

Ans. Altitude $= r\sqrt{2}$.

16. Find the altitude of the cone of maximum convex surface that can be inscribed in a given sphere whose radius is r.

Ans. Altitude $= 4r/3$.

17. A privateer has to pass between two lights, A and B, on opposite headlands. The intensity of each light is known, and also the distance between them. At what point must the privateer cross the line joining the lights so as to be in the light as little as possible?

Let $c =$ the distance AB,
and $x =$ the distance from A to any point P on AB.

Let a and b be the intensities of the lights A and B, respectively, at a unit's distance. The intensity of a light at any point equals its intensity at a unit's distance divided by the square of the distance of the point from the light.

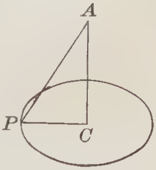
Hence, the function whose minimum we seek is

$$a/x^2 + b/(c-x)^2.$$

Ans. $x = ca^{1/3}/(a^{1/3}+b^{1/3})$.

18. The flame of a lamp is directly over the centre of a circle whose radius is r; what is the distance of the flame above the centre when the circumference is illuminated as much as possible?

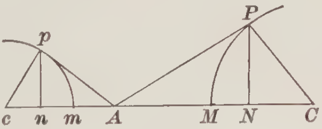
Let A be the flame, P any point on the circumference, and $x = AC$. The intensity of illumination at P varies directly as sin CPA, and inversely as the square of PA. Hence, the function whose maximum is required is $ax/(r^2+x^2)^{3/2}$, where r is the radius of the circle, and a is the intensity of illumination at a unit's distance from the flame.



Ans. $x = r\sqrt{2}/2$.

19. On the line joining the centres of two spheres, find the point from which the maximum of spherical surface is visible.

Let $cp = r$, $CP = R$, $cC = a$, and $cA = x$, A being any point on mM. From A draw the tangents Ap and AP; then the sum of the zones whose altitudes are nm and NM, respectively, is the function whose maximum is required.



By geometry this function is

$$2\pi\left[r^2 + R^2 - \left(\frac{r^3}{x} + \frac{R^3}{a-x}\right)\right].$$

Ans. $x = ar^{3/2}/(r^{3/2}+R^{3/2})$.

20. Assuming that the work of driving a steamer through the water varies as the cube of her speed, show that her most economical rate per hour against a current running c miles per hour is $3c/2$ miles per hour.

Let $v =$ the speed of the steamer in miles per hour.

Then $av^3 =$ the work per hour, a being a constant;

and $v - c =$ the actual distance advanced per hour.

Hence, $av^3/(v-c) =$ the work per mile of actual advance.

21. The amount of fuel consumed by a certain ocean steamer varies as the cube of her speed. When her speed is 15 miles per hour she consumes $4\frac{1}{2}$ tons of coal per hour at \$4 per ton. The other expenses are \$12 per hour. Find her most economical speed and the minimum cost of a voyage of 2080 miles. *Ans.* 10.4 miles per hour; \$3600.

22. Find the parabola of minimum area which shall circumscribe a given circle whose radius is r. *Ans.* $y^2 = rx$.

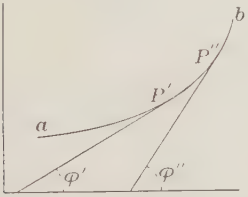
23. One dark night the captain of a man-of-war saw a privateersman crossing his path at right angles and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could make only b miles in the same time. The captain's only hope was to cross the track of the privateersman at as short a distance as possible under his stern, and to disable him by one or two well-directed shots; so the ship's lights were put out and her course altered so as to effect this. Show that the man-of-war crossed the privateersman's track $(c/b)\sqrt{(a^2-b^2)}$ miles astern of the latter.

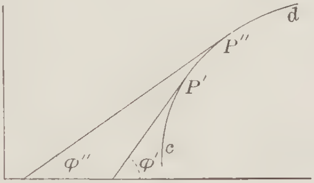
24. The limited line AB lies without and is oblique to the indefinite line CD; find the point P in CD so that the angle APB will be a maximum. *Ans.* If AB produced meets CD in C, $PC = \sqrt{AC \cdot BC}$.

CHAPTER VIII.

POINTS OF INFLEXION. CURVATURE. EVOLUTES.

108. *A curve is concave upward or downward at any point* (x, y) *according as* d^2y/dx^2 *is positive or negative.*

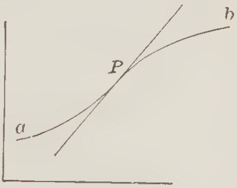




When a curve, as *ab*, is concave upward, tan ϕ or dy/dx evidently increases when x increases; hence, by Cor. of § 80, d^2y/dx^2 is positive.

When a curve, as *cd*, is concave downward, tan ϕ or dy/dx evidently decreases when x increases; hence, d^2y/dx^2 is negative.

109. A **point of inflexion** is a point, as P, where the tangent crosses the curve at the point of contact.



On opposite sides of a point of inflexion, as P, the curve is concave in opposite directions, and d^2y/dx^2 has opposite signs; hence, at a point of inflexion dy/dx has either a maximum or a minimum value (§ 101).

Therefore, to examine a curve for points of inflexion, we examine its slope dy/dx for maxima and minima.

The road or path whose grade is aPb is steepest at the point of inflexion, P.

EXAMPLES.

1. Examine $y = a + c(x+b)^3$ for points of inflexion.

Here $d^2y/dx^2 = 6c(x+b)$.

The root of $6c(x+b) = 0$

is $-b$, and $6c(x+b)$ evidently changes from $-$ to $+$ when x passes through $-b$; hence, $(-b, a)$ is a point of inflexion, or a point of minimum slope. To the right of $(-b, a)$ the curve is concave upward.

2. Examine $x^3 - 3bx^2 + a^2y = 0$ for points of inflexion.

Ans. $(b, 2b^3/a^2)$ is a point of inflexion, or of maximum slope, to the right of which the curve is concave downward.

3. Examine $y = x^3 - 3x^2 - 9x + 9$ for points of inflexion.

Ans. $(1, -2)$ is a point of inflexion, to the right of which the curve is concave upward.

4. Examine $y = c\sin(x/a)$ for points of inflexion.

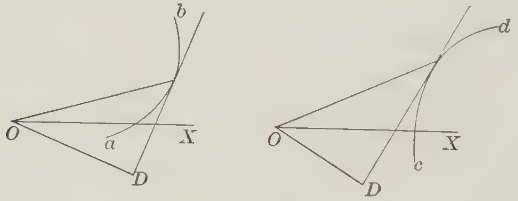
Ans. $(0, 0)$, $(\pm a\pi, 0)$, $(\pm 2a\pi, 0)$, $\cdots$.

5. Examine the witch of Agnesi $y = 8a^3/(x^2 + 4a^2)$ for points of inflexion. *Ans.* $(\pm 2a\sqrt{3}/3, 3a/2)$.

6. Examine the curve $y = x^3/(a^2 + x^2)$ for points of inflexion.

Ans. $(0, 0)$, $(a\sqrt{3}, 3a\sqrt{3}/4)$, $(-a\sqrt{3}, -3a\sqrt{3}/4)$.

110. Polar curves. From the figure it is evident that when a polar curve, as ab, is concave toward the pole, p or OD increases as ρ increases; hence, $dp/d\rho$ is positive.



When a curve, as cd, is convex toward the pole, p decreases as ρ increases; hence, $dp/d\rho$ is negative.

That is, *a polar curve is concave or convex toward the pole according as* $dp / d\rho$ *is positive or negative.*

At a point of inflexion on a polar curve, $dp / d\rho$ changes its quality, and therefore p is a maximum or a minimum; and conversely. Hence, to examine a polar curve for points of inflexion, we examine p for maxima and minima.

Ex. Examine the lituus $\rho^2\theta = a^2$ for points of inflexion.

Here $$p = \frac{\rho^2}{\sqrt{\rho^2 + (d\rho / d\theta)^2}} = \frac{2\,a^2\rho}{\sqrt{4\,a^4 + \rho^4}}. \qquad \text{§ 63, (9)}$$

$$\therefore \frac{dp}{d\rho} = \frac{2\,a^2(4\,a^4 - \rho^4)}{(4\,a^4 + \rho^4)^{3/2}}.$$

Hence, $\rho = a\sqrt{2}$ renders p a maximum; therefore, $(a\sqrt{2}, 1/2)$ is a point of inflexion.

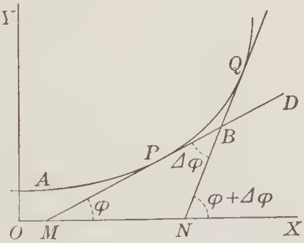
In the logarithmic spiral $\rho = a^\theta$, $dp / d\rho$ is always positive; hence, the curve, being concave toward the pole at all points, has no point of inflexion.

111. The **curvature** of any curve, as APQ (§ 112, fig.), at any point, as P, is the s-rate at which the curve bends at P, or the s-rate at which the tangent revolves, where s denotes the length of the variable arc AP.

112. *The curvature of a curve at* (x, y) *is* $d\phi / ds$ *radians to a unit of* s.

Let $AP = s$, and let ϕ denote (in radians) the variable angle XMP as P moves along the curve APQ; then, evidently, the curvature of APQ at P equals the s-rate of ϕ, or

$$d\phi / ds.$$



113. Curvature of a circle. Let APQ be the arc of a circle whose radius is r; then the angle MBN, or $\Delta\phi$, will equal the angle subtended by PQ, or Δs, at its centre.

Hence, by § 37, we have

$$\Delta\phi = \Delta s / r; \quad \therefore d\phi / ds = 1 / r. \qquad \text{§ 11}$$

That is, *the curvature of a circle is constant, and equals* 1 / r *radians to a unit of arc.*

For example, if $r = 5$, the circle bends uniformly at the rate of 1 / 5 radian to a unit of arc.

If $r = 1/3$, the curvature of the circle is 3 radians per unit of arc.

114. Circle of curvature. The curvature of any curve except the circle varies from one point to another. A circle tangent to a curve and having the same curvature as the curve at the point of contact is called the *circle of curvature* at that point; its radius is called the *radius of curvature;* and its centre, the *centre of curvature.*

Let R denote the radius of the circle of curvature at any point of a curve; then, since the curvature of the curve, or $d\phi / ds$, equals the curvature of the circle, we have

$$d\phi / ds = 1 / R, \text{ or } R = ds / d\phi.$$

If at P (§ 112, fig.) the *direction* of the path of (x, y) became *constant*, (x, y) would trace the tangent at P; if at P the *change* of direction of the path became *uniform* with respect to s, (x, y) would trace the circle of curvature at P.

115. *To find* R *in terms of* x *and* y.

$$ds / dx = [1 + (dy / dx)^2]^{1/2}. \qquad \text{§ 33, Cor. 1}$$

$$\phi = \tan^{-1} (dy / dx); \qquad \text{§ 33}$$

$$\therefore \frac{d\phi}{dx} = \frac{d^2y / dx^2}{1 + (dy / dx)^2}.$$

$$\therefore R = \frac{ds}{d\phi} = \frac{[1 + (dy / dx)^2]^{3/2}}{d^2y / dx^2}. \qquad (1)$$

R will be positive or negative according as d^2y / dx^2 is positive or negative; that is, according as the curve is concave upward or downward.

If we take the reciprocals of the members of (1), we obtain the curvature.

EXAMPLES.

Find R and the curvature of each of the following curves:

1. The parabola $y^2 = 4px$.

$$dy/dx = 2p/y, \quad d^2y/dx^2 = -4p^2/y^3.$$

Substituting these values in (1) of § 115, we obtain

$$R = \left(\frac{y^2 + 4p^2}{y^2}\right)^{3/2} \frac{y^3}{4p^2} = \frac{2(x+p)^{3/2}}{p^{1/2}}.$$

We neglect the quality of R, since the quality of d^2y/dx^2 indicates whether the curve is concave upward or downward.

At the vertex (0, 0), $R = 2p$, and the curvature is $(1/2p)$ radian to a unit of arc.

2. The equilateral hyperbola $2xy = a^2$. $\qquad R = (x^2 + y^2)^{3/2}/a^2$.

3. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. $\qquad \dfrac{d\phi}{ds} = \dfrac{a^4b^4}{(a^4y^2 + b^4x^2)^{3/2}}$.

The maximum curvature is a/b^2, and the minimum b/a^2.

4. The curve $y = x^4 - 4x^3 - 18x^2$ at (0, 0). $\quad R = 1/36$.

5. The logarithmic curve $y = a^x$. $\qquad \dfrac{d\phi}{ds} = \dfrac{my}{(m^2 + y^2)^{3/2}}$.

6. The cubical parabola $y^3 = a^2x$. $\qquad \dfrac{d\phi}{ds} = \dfrac{6a^4y}{(9y^4 + a^4)^{3/2}}$.

7. The cycloid $x = r \operatorname{vers}^{-1}(y/r) \mp \sqrt{2ry - y^2}$. $\qquad R = 2\sqrt{2ry}$.

At the highest point $R = 4r$, or the maximum of R is $4r$.

8. The catenary $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$. $\qquad \dfrac{d\phi}{ds} = \dfrac{a}{y^2}$.

9. The hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$. $\qquad R = 3(axy)^{1/3}$.

10. The curve $x^{1/2} + y^{1/2} = a^{1/2}$. $\qquad \dfrac{d\phi}{ds} = \dfrac{a^{1/2}}{2(x+y)^{3/2}}$.

11. The curve $y^3 = 6x^2 + x^3$. $\qquad \dfrac{d\phi}{ds} = \dfrac{8x^2y}{[y^4 + (4x + x^2)^2]^{3/2}}$.

116. *To find* R *in terms of* ρ *and* θ.

From (4) and (10) of § 63, we have

$$\phi = \theta + \psi, \qquad \psi = \tan^{-1}\frac{\rho}{d\rho/d\theta}.$$

$$\therefore \frac{d\phi}{d\theta} = 1 + \frac{d\psi}{d\theta}, \quad \frac{d\psi}{d\theta} = \frac{(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}{\rho^2 + (d\rho/d\theta)^2}.$$

$$\therefore \frac{d\phi}{d\theta} = \frac{\rho^2 + 2(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}{\rho^2 + (d\rho/d\theta)^2}.$$

$$\therefore R = \frac{ds/d\theta}{d\phi/d\theta} = \frac{[\rho^2 + (d\rho/d\theta)^2]^{3/2}}{\rho^2 + 2(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}. \quad \text{§ 63, (3)}$$

EXAMPLES.

Find R in each of the following curves:

1. The spiral of Archimedes $\rho = a\theta$.

Here $d\rho/d\theta = a, \quad d^2\rho/d\theta^2 = 0;$

$$\therefore R = \frac{(\rho^2 + a^2)^{3/2}}{\rho^2 + 2a^2} = \frac{a(\theta^2+1)^{3/2}}{\theta^2 + 2}.$$

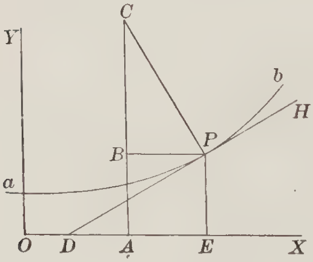
2. The logarithmic spiral $\rho = a^\theta$. $R = \rho\sqrt{1 + (\log a)^2}$.

3. The lemniscate $\rho^2 = a^2 \cos 2\theta$. $R = a^2/3\rho$.

4. The cardioid $\rho = a(1 - \cos\theta)$. $R = 2\sqrt{2a\rho}/3$.

5. The curve $\rho = a\sec^2(\theta/2)$. $R = 2a\sec^3(\theta/2)$.

117. Co-ordinates of centre of curvature. Let $P(x, y)$ be any point on the curve ab, and $C(\alpha, \beta)$ the corresponding centre of curvature.



Then PC equals R and is perpendicular to the tangent PD.

Hence, $\angle BCP = \angle XDP$
$= \phi.$

$OA = OE - BP,$
$AC = EP + BC;$

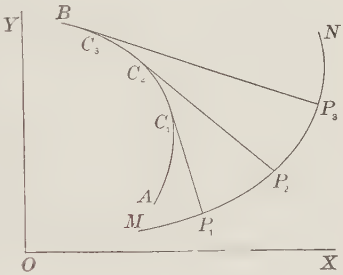
that is, $$\alpha = x - R\sin\phi,\quad \beta = y + R\cos\phi; \tag{1}$$

or $$\alpha = x - R\frac{dy}{ds},\qquad \beta = y + R\frac{dx}{ds}. \tag{2}$$

Substituting in (2) the values of R and ds, we have

$$\alpha = x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{d^2y/dx^2},\qquad \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}. \tag{3}$$

118. Evolutes and involutes. If the point (x, y) moves along the curve MN, by equations (3) of § 117 the point (α, β) will trace some other curve, as AB. The curve AB, which is the locus of the centres of curvature of MN, is called the *evolute* of MN.



To express the inverse relation, MN is called the *involute* of AB.

119. Properties of the involute and evolute.

I. From Cor. 1 of § 33 and $ds = R\,d\phi$, we have

$$dx = \cos\phi\, ds = R\cos\phi\, d\phi, \tag{1}$$

and $$dy = \sin\phi\, ds = R\sin\phi\, d\phi. \tag{2}$$

Differentiating equations (1) of § 117, and using the relations given in (1) and (2), we obtain

$$d\alpha = dx - R\cos\phi\, d\phi - \sin\phi\, dR = -\sin\phi\, dR, \tag{3}$$

$$d\beta = dy - R\sin\phi\, d\phi + \cos\phi\, dR = \cos\phi\, dR. \tag{4}$$

Dividing (4) by (3), we obtain

$$d\beta/d\alpha = -\cot\phi = -dx/dy.$$

That is, *the normal to the involute at* (x, y), *as* P_1 (§ 118, fig.), *is tangent to the evolute at the corresponding point* (α, β), *as* C_1.

II. Squaring and adding (3) and (4), we obtain

$$d\alpha^2 + d\beta^2 = dR^2.$$

Let s denote the length of an arc of the evolute; then

$$d\alpha^2 + d\beta^2 = ds^2.$$

Hence, $$ds = \pm\, dR; \quad \therefore \Delta s = \pm\, \Delta R.$$

That is, *the difference between two radii of curvature, as* C_3P_3 *and* C_1P_1 (§ 118, fig.), *is equal to the corresponding arc of the evolute, as* C_1C_3.

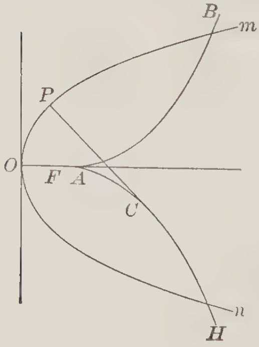
These two properties show that the involute MN can be traced by a point in a string unwound from the evolute AB. From this property the *evolute* receives its name.

120. *To find the equation of the evolute of a given involute.* Differentiating the equation of the involute and using equations (3) of § 117, we obtain α and β in terms of x and y. These two equations and that of the involute form a system of three equations between α, β, x, and y.

Eliminating x and y from these equations, we obtain a relation between α and β, or the equation sought.

EXAMPLES.

Find the equation of the evolute of



1. The parabola $y^2 = 4px$. (1)

$$dy/dx = 2p/y, \quad d^2y/dx^2 = -4p^2/y^3.$$

Substituting these values in (3) of § 117, we obtain

$$\alpha = 3x + 2p; \qquad \beta = -y^3/4p^2;$$

$$\therefore x = (\alpha - 2p)/3, \quad y = -\sqrt[3]{4\beta p^2}.$$

Substituting these values of x and y in (1), we obtain

$$\beta^2 = 4(\alpha - 2p)^3/27p, \tag{2}$$

as the equation of the evolute of (1).

The locus of (2) is the semi-cubical parabola. Thus, if *nom* is the locus of (1), F being the focus, then HAB is the locus of (2), where $OA = 2p = 2 \cdot OF$.

2. The ellipse $a^2y^2 + b^2x^2 = a^2b^2$. (1)

Here $$x^2 = \left(\frac{a^4\alpha}{a^2 - b^2}\right)^{2/3}, \quad y^2 = \left(\frac{b^4\beta}{a^2 - b^2}\right)^{2/3}.$$

Hence, the equation of the evolute of (1) is

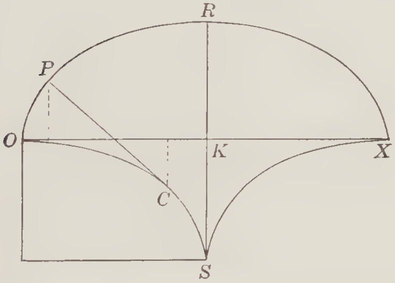
$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}. \qquad \text{§ 155, fig. 9}$$

3. The cycloid $x = r \text{ vers}^{-1}(y/r) \mp \sqrt{2ry - y^2}$.

$$\frac{dy}{dx} = \frac{\sqrt{2ry - y^2}}{y}, \quad \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

$$\therefore y = -\beta, \quad x = \alpha - 2\sqrt{-2r\beta - \beta^2}.$$

$$\therefore \alpha = r \text{ vers}^{-1}(-\beta/r) \pm \sqrt{-2r\beta - \beta^2}. \qquad (1)$$



The locus of (1) is another cycloid equal to the given cycloid, the highest point being at the origin.

That is, *the evolute of a cycloid is an equal cycloid.*

Thus, the evolute of the arc OR is the arc OS, which equals RX; and the evolute of RX is SX, which equals OR.

4. The hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

5. Find the length of one branch of the cycloid.

Here $$R = 2\sqrt{2ry}; \quad \therefore SR = 4r.$$

$$ORX = 2 \cdot OS = 2 \cdot SR = 8r.$$

6. The length of the evolute of the ellipse is $4(a^3 - b^3)/ab$.

Find four times the difference between R at $(0, b)$ and R at $(a, 0)$.

7. Find the length of an arc of the evolute of the parabola $y^2 = 4px$ in terms of the abscissas of its extremities.

$$\text{Arc } AC = CP - AO = \frac{2(x+p)^{3/2}}{\sqrt{p}} - 2p \qquad \text{Example 1, fig.}$$

$$= \frac{2}{\sqrt{p}}\left(\frac{\alpha + p}{3}\right)^{3/2} - 2p.$$

8. Show that in the catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$,

$$\alpha = x - \frac{y}{a}\sqrt{y^2 - a^2}, \quad \beta = 2y.$$

9. Find the centre of curvature, and the equation of the evolute, of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.

Ans. $\alpha = x + 3x^{1/3}y^{2/3}$; $\beta = y + 3x^{2/3}y^{1/3}$;
$(\alpha + \beta)^{2/3} + (\alpha - \beta)^{2/3} = 2a^{2/3}$.

CHAPTER IX.

ENVELOPES. ORDER OF CONTACT. OSCULATING CURVES.

121. Family of curves. If in the equation,

$$f(x, y, a) = 0,$$

different values are assigned to a, the resulting equations will represent a series of curves differing in position or form, but all belonging to the same class or family of curves.

For example, if different values are assigned to α in the equation,

$$(x - \alpha)^2 + y^2 = r^2,$$

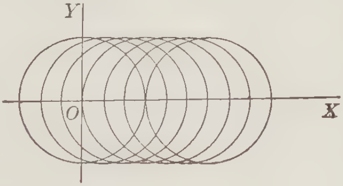
the loci of the resulting equations will be a series of circles all having their centres on the x-axis and the same radius r (§ 122, fig.).

As used in this chapter, the word curve includes the straight line.

The quantity a, which is constant for the same curve, but different for different curves, is called the *parameter* of the family. Any two curves of the family which correspond to nearly equal values of a are called *consecutive* curves.

Of the families considered in this chapter the consecutive curves intersect.

122. Envelopes. If when consecutive curves approach indefinitely near each other, their points of intersection approach limits, the locus of these limits is called the *envelope* of the family.



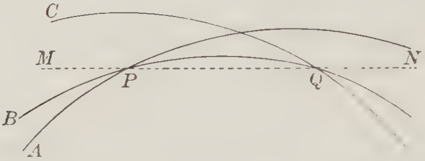
For example, if α is a variable parameter, the envelope of the family of curves represented by the equation,

$$(x - \alpha)^2 + y^2 = 25,$$

is evidently the two lines $y = \pm 5$.

This envelope is evidently tangent to each curve of the family.

123. *The envelope is tangent to each curve of the family.* Let A, B, C represent three consecutive curves of the family. Let P be the point of intersection of the curves A and B, and Q that of B and C. Conceive the curves A and C to approach the curve B, so that arc $PQ \doteq 0$; then, since the limits of P and Q are on both the envelope and the curve B, the limit of the secant $MPQN$ will be a common tangent to the envelope and the curve B. Hence, the envelope is tangent to the curve B at their common point.



124. *To find the equation of the envelope of a family of curves.*

Let
$$f(x, y, a) = 0, \tag{1}$$
and
$$f(x, y, a + \Delta a) = 0, \tag{2}$$
be the equations of any two consecutive curves.

Subtracting (1) from (2) and dividing by Δa, we have

$$\frac{f(x, y, a + \Delta a) - f(x, y, a)}{\Delta a} = 0. \tag{3}$$

By algebra the intersections of (1) and (3) are the same as those of (1) and (2).

Making $\Delta a \doteq 0$, from (3) we obtain (For notation see § 133)

$$\frac{\partial}{da} f(x, y, a) = 0. \tag{4}$$

The intersections of (1) and (4) are, therefore, the *limits* of the intersections of (1) and (2). Eliminating a between (1) and (4), we obtain the equation of the envelope.

EXAMPLES.

1. Find the envelope of the family of straight lines represented by the equation $y = \alpha x + m/\alpha$. (1)

Differentiating (1) with respect to α, we have

$$0 = x - m/\alpha^2. \tag{2}$$

Eliminating α between (1) and (2), we obtain

$$y^2 = 4\,mx\,; \tag{3}$$

that is, the envelope is the parabola (3).

2. Find the envelope of the hypotenuse of the right-angled triangles which have the constant area c.

Let α and β denote the lengths of the sides of the right triangles, and assume these sides as the co-ordinate axes; then we have

$$x/\alpha + y/\beta = 1, \quad \alpha\beta = 2\,c. \tag{1}$$

Eliminating β between equations (1), we obtain

$$x/\alpha + \alpha y/2\,c = 1. \tag{2}$$

Differentiating (2) with respect to α, we have

$$-\,x/\alpha^2 + y/2\,c = 0. \tag{3}$$

Eliminating α between (2) and (3), we obtain

$$xy = c/2\,;$$

that is, the envelope is an hyperbola to which the sides of the triangle are asymptotes.

3. Find the envelope of a line of constant length c whose extremities move along two fixed rectangular axes.

Let α and β be the intercepts of the line on the axes; then

$$x/\alpha + y/\beta = 1, \qquad \alpha^2 + \beta^2 = c^2. \tag{1}$$

Differentiating equations (1), we obtain

$$-\frac{x}{\alpha^2} = \frac{y}{\beta^2}\frac{d\beta}{d\alpha}, \qquad -\,\alpha = \beta\,\frac{d\beta}{d\alpha}. \tag{2}$$

Dividing the first of equations (2) by the second, we have

$$\frac{x/\alpha}{\alpha^2} = \frac{y/\beta}{\beta^2} = \frac{x/\alpha + y/\beta}{\alpha^2 + \beta^2} = \frac{1}{c^2}.$$

$$\therefore \alpha = (xc^2)^{1/3}, \qquad \beta = (yc^2)^{1/3}.$$

Substituting these values in either of equations (1), we find the envelope to be the hypocycloid $x^{2/3} + y^{2/3} = c^{2/3}$. (§ 155, fig. 8.)

4. Find the envelope of the family of lines whose equation is

$$x/\alpha + y/\beta = 1,$$

α and β having the relation $\alpha/l + \beta/m = 1$.

Ans. $(x/l)^{1/2} + (y/m)^{1/2} = 1$.

5. Find the envelope of the family of ellipses defined by the equations

$$x^2/\alpha^2 + y^2/\beta^2 = 1, \qquad \alpha^2/m^2 + \beta^2/n^2 = 1.$$

Ans. The four lines $\pm x/m \pm y/n = 1$.

6. Find the envelope of the family of right lines,

$$y = \alpha x \pm \sqrt{a^2\alpha^2 - b^2}, \tag{1}$$

where α is the variable parameter.

Here $$0 = x \pm \frac{a^2\alpha}{\sqrt{a^2\alpha^2 - b^2}}; \quad \therefore \alpha = \mp \frac{bx}{a\sqrt{x^2 - a^2}}. \tag{2}$$

Substituting this value of α in (1), we obtain

$$y = \mp \frac{b}{a} \frac{(x^2 - a^2)}{\sqrt{x^2 - a^2}} = \mp \frac{b}{a} \sqrt{x^2 - a^2},$$

or $$x^2/a^2 - y^2/b^2 = 1. \tag{3}$$

In equations (1) and (2) the upper signs go together.

Here, as in example 1, the equation of the tangent is given and that of the curve is required; hence, the method of envelopes has sometimes been called "the inverse method of tangents."

7. Find the envelope of the family of parabolas $y^2 = \alpha(x - \alpha)$, α being a variable parameter. *Ans.* $y = \pm x/2$.

8. Find the envelope of the family of circles whose diameters are the double ordinates of the parabola $y^2 = 4px$. *Ans.* $y^2 = 4p(p + x)$.

9. Find the envelope of the family of circles whose diameters are the double ordinates of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Ans. $x^2/(a^2 + b^2) + y^2/b^2 = 1$.

10. Show that the envelope of the normals to any curve, MN (§ 118, fig.), is the evolute AB of that curve.

Let P_2 approach P_3 as its limit; then the intersection of the normals P_2C_2 and P_3C_3 will evidently approach C_3 as its limit.

Hence, the evolute AB is the envelope of the normals to MN.

11. Using the principle in example 10, find the evolute of the parabola $y^2 = 4px$, having given the equation of the normal in the form

$$y = \alpha(x - 2p) - p\alpha^3.$$

12. Find the envelope of the family of ellipses

$$x^2/\alpha^2 + y^2/(k-\alpha)^2 = 1,$$

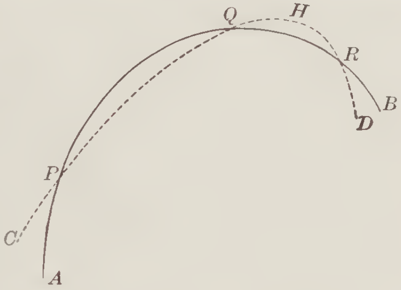
α being a variable parameter. *Ans.* $x^{2/3} + y^{2/3} = k^{2/3}$.

13. Find the envelope of the family of parabolas

$$y = \alpha x - (1 - \alpha^2)x^2/2p,$$

α being a variable parameter. *Ans.* $x^2 = -p(p + 2y)$.

125. Different orders of contact. Let APB and CPD be any two curves, and let P, Q, R be three of their points of intersection. Suppose that Q moves toward P and becomes coincident with it. The curves are then said to have *contact of the first order* at P.



At such a point the curves do not cross each other, since CP and QH are on the same side of APB.

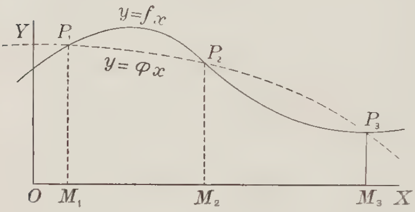
Again, suppose that Q and R both become coincident with P. Three points of intersection will then coincide at P, and the curves are said to have *contact of the second order.*

At such a point the curves cross each other, since CP and RD are on opposite sides of APB.

If four points of intersection become coincident at P, the *contact* is of the *third order,* and the curves do not cross at P.

In general, *if* (k + 1) *points of intersection coincide at* P, *the contact is said to be of the* kth *order, and the curves will, or will not, cross at* P *according as the order of contact is even or odd.*

126. Analytic conditions for contact of the kth order. Let the curves $y = fx$ and $y = \phi x$ intersect in the points P_1, P_2, P_3, $\cdots$.



Let $OM_1 = a$, and let h denote the distances from M_1 to the ordinates M_1P_1, M_2P_2, M_3P_3, $\cdots$; that is, let $h = 0$, M_1M_2, M_1M_3, $\cdots$.

Then, for each of these values of h, we have

$$\phi(a+h) = f(a+h), \text{ or } 0 = f(a+h) - \phi(a+h). \quad (1)$$

Expanding the second member of (1) by Taylor's theorem, we have

$$0 = (fa - \phi a) + (f'a - \phi' a)h + (f''a - \phi'' a)\frac{h^2}{\lfloor 2}$$
$$+ (f'''a - \phi''' a)\frac{h^3}{\lfloor 3} + \cdots + (f^k a - \phi^k a)\frac{h^k}{\lfloor k} + \cdots. \quad (2)$$

If P_2 coincides with P_1, that is, if two values of h are zero; by the theory of equations, from (2) we have

$$fa = \phi a, \qquad f'a = \phi' a. \quad (3)$$

Hence, (3) are the two conditions for contact of the first order.

If three values of h are zero, from (2) we have

$$fa = \phi a, \qquad f'a = \phi' a, \qquad f''a = \phi'' a. \quad (4)$$

Hence, (4) are the three conditions for contact of the second order.

In general, if $k + 1$ values of h are zero, from (2) we have

$$fa = \phi a; \quad f'a = \phi' a, \quad f''a = \phi'' a, \quad \cdots, \quad f^k a = \phi^k a. \quad (5)$$

Hence, (5) are the $k + 1$ conditions for contact of the kth order.

Or, in the differential notation, if when $x = a$

$$y, \quad dy/dx, \quad d^2y/dx^2, \quad \cdots, \quad d^ky/dx^k$$

all have the same values in one of two equations as in the other, the loci of these equations have contact of the kth order at (a, y).

Ex. Find the values of a, b, c when the curves

$$y = ax^2 + bx + c \text{ and } y = \log(x - 3)$$

have contact of the second order at the point (4, 0).

From $y = \log(x - 3)$, we have when $x = 4$,

$$y = 0, \qquad dy/dx = 1, \qquad d^2y/dx^2 = -1. \tag{1}$$

From $y = ax^2 + bx + c$, we have when $x = 4$,

$$y = 16a + 4b + c, \quad dy/dx = 8a + b, \quad d^2y/dx^2 = 2a. \tag{2}$$

Equating the values of y, dy/dx, and d^2y/dx^2 in (1) and (2), we obtain

$$16a + 4b + c = 0, \quad 8a + b = 1, \quad 2a = -1. \tag{3}$$

Solving system (3), we obtain

$$a = -1/2, \quad b = 5, \quad c = -12.$$

Hence, the curve $y = -x^2/2 + 5x - 12$ has contact of the second order with the curve $y = \log(x - 3)$ at the point (4, 0).

Only three independent conditions can be imposed on the three general constants a, b, c; hence, in general, the given curves cannot have an order of contact above the second.

127. Osculating curves. The *straight line of closest contact* (a tangent) has, in general, contact of the first order; for two and only two independent conditions can be imposed upon the two arbitrary constants in the general linear equation,

$$y = mx + c.$$

The *circle of closest contact*, called the *osculating circle*, has, in general, contact of the second order; for three and only three independent conditions can be imposed upon the three arbitrary constants in the general equation of the circle,

$$(x - a)^2 + (y - b)^2 = r^2.$$

The *parabola of closest contact*, called the *osculating parabola*, has, in general, contact of the third order; for the general equation of the parabola has four arbitrary constants.

The *conic of closest contact*, called the *osculating conic*, has, in general, contact of the fourth order; for the general equation of the conic has five arbitrary constants.

It was necessary to qualify the above propositions by the words 'in general'; for at particular points the contact may be of a higher order than at points in general. For example, at a point of maximum or minimum slope the tangent has three coincident points in common with the curve, the contact is of the second order, and the tangent crosses the curve (§ 109). Again, at a point of maximum or minimum curvature, the circle of curvature, which does not cross the curve at the point of contact (§ 130), has contact of the third order at least.

128. *The osculating circle is the circle of curvature.*

From §§ 115 and 126 it follows that any two curves which have contact of the second order have the same curvature at their common point; and conversely.

129. *The circle of curvature, in general, crosses the curve at the point of contact.*

For its contact is, in general, of an even order.

130. **At a point of maximum or minimum curvature, the circle of curvature does not cross the curve.*

For on each side of a point of maximum curvature, the curve changes its direction more slowly than at this point; hence, on each side of this point, the curve lies without the circle of curvature at this point, and therefore does not cross it.

For a similar reason, the circle of curvature at a point of minimum curvature does not cross the curve.

Since the circle of curvature at a point of maximum or minimum curvature does not cross the curve, the contact must be of an odd order (the third at least).

---

* By a maximum or a minimum curvature is meant an arithmetic maximum or minimum, the quality of the curvature not being considered.

EXAMPLES.

1. By drawing the figures, show that the four intersections of a circle and an ellipse coincide when the circle becomes the osculating circle to the ellipse at either end of the major or the minor axis.

2. Show that the curve $4y = 3x^2 - x^3$ and the line $4y = 3x - 1$ have contact of the second order.

3. Show that the parabola $8y = x^2 - 8$ and the circle $x^2 + y^2 = 6y + 7$ have contact of the third order.

4. Find the order of contact of the hyperbola $xy = 1$ and the parabola $(x - 2)^2 + (y - 2)^2 = 2xy$.

5. Find the value of a when the hyperbola $xy = 3x - 1$ and the parabola $y = x + 1 + a(x - 1)^2$ have contact of the second order.

Ans. $a = -1$.

6. Find the values of m and c when $y = mx + c$ has contact of the second order with $y = x^3 - 3x^2 - 9x + 9$. *Ans.* $m = -12$, $c = 10$.

7. Find the values of a, b, c when the curves

$$y = x^3 \text{ and } y = ax^2 + bx + c$$

have contact of the second order at the point (1, 1).

8. Find the values of a, b, c when the curves

$$y = \sin x \text{ and } y = ax^2 + bx + c$$

have contact of the third order at $(\pi/2, 1)$.

Ans. $a = -1/2$, $b = \pi/2$, $c = 1 - \pi^2/8$.

CHAPTER X.

CHANGE OF THE INDEPENDENT VARIABLE.

131. Different forms of the successive derivatives of dy / dx.

(i) When x is independent, by § 78 we have

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}, \quad \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d^3y}{dx^3}, \cdots.$$

(ii) When neither x nor y is independent, dy/dx is a fraction having a variable numerator and a variable denominator, and $d\,dx = d^2x$, etc.; hence,

$$\frac{d}{dx}\frac{dy}{dx} = \frac{dx\,d^2y - dy\,d^2x}{dx^3}, \tag{1}$$

$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{dx^2d^3y - dx\,dy\,d^3x - 3\,dx d^2x d^2y + 3\,dy\,(d^2x)^2}{dx^5}, \tag{2}$$

. . . .

(iii) When y is independent, $d^2y = 0$, $d^3y = 0$, $\cdots$; hence, from (1) and (2) we obtain

$$\frac{d}{dx}\frac{dy}{dx} = -\frac{dy\,d^2x}{dx^3}, \tag{1'}$$

$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{3\,dy\,(d^2x)^2 - dx\,dy\,d^3x}{dx^5}, \tag{2'}$$

. . . .

132. Change of the independent variable. In the applications of the Calculus it is sometimes necessary to make a differential equation depend on a new independent variable instead of the one which was originally selected; that is, we need *to change the independent variable.*

When $x = \phi(z)$ and we wish to change the independent variable from x to z, we substitute for d^2y/dx^2, d^3y/dx^3, $\cdots$, respectively, the second members of (1), (2), $\cdots$, in § 131.

In the resulting equation we substitute for x, dx, d^2x, $\cdots$, their values as obtained from the equation $x = \phi(z)$.

Ex. 1. Given $x = \cos\theta$, change the independent variable from x to θ in

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0. \tag{1}$$

Substituting for d^2y/dx^2 the second member of (1) in § 131, we have

$$\frac{dx\,d^2y - dy\,d^2x}{dx^3} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0. \tag{2}$$

$$x = \cos\theta, \qquad \therefore\ dx = -\sin\theta d\theta,$$

$$d^2x = -\cos\theta d\theta^2, \quad 1 - x^2 = \sin^2\theta.$$

Substituting these values in (2) and simplifying, we obtain

$$d^2y/d\theta^2 + y = 0. \tag{3}$$

Equation (3) is the differential equation which would have been obtained if the differentiations which led to (1) when x was independent had been performed on the same equation under the new hypothesis that $x = \cos\theta$ and θ is independent.

Ex. 2. Given $y = \tan z$, change the function from y to z in

$$\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2. \tag{1}$$

$$y = \tan z, \qquad \therefore\ \frac{dy}{dx} = \sec^2 z\,\frac{dz}{dx},$$

$$\frac{d^2y}{dx^2} = 2\sec^2 z \tan z\left(\frac{dz}{dx}\right)^2 + \sec^2 z\,\frac{d^2z}{dx^2},$$

$$1 + y = 1 + \tan z, \quad 1 + y^2 = \sec^2 z.$$

Substituting these values in (1), we obtain

$$d^2z/dx^2 - 2(dz/dx)^2 = \cos^2 z.$$

To change the independent variable from x to y we substitute for d^2y/dx^2, d^3y/dx^3, $\cdots$, respectively, the last members of (1′), (2′), $\cdots$, in § 131.

Ex. 3. Change the independent variable from x to y in

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0. \qquad (1)$$

Substituting in (1) for d^2y/dx^2 and d^3y/dx^3, respectively, the last members of (1′) and (2′) in § 131, we obtain an equation which may be reduced to the form

$$d^3x/dy^3 + d^2x/dy^2 = 0. \qquad (2)$$

The position of dy in (2) indicates that y is independent.

EXAMPLES.

Change the independent variable from x to z in

1. $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, given $x = e^z$.
Ans. $d^2y/dz^2 + (a-1)(dy/dz) + by = 0$.

2. $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{a^2}{x^2}y = 0$, given $x = \frac{1}{z}\cdot$ *Ans.* $\frac{d^2y}{dz^2} + a^2y = 0$.

3. $(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$, given $x = \cos z$. *Ans.* $\frac{d^2y}{dz^2} = 0$.

4. $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$, given $x = \tan z$.
Ans. $d^2y/dz^2 + y = 0$.

5. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$, given $x^2 = 4z$. *Ans.* $z\frac{d^2y}{dz^2} + \frac{dy}{dz} + y = 0$.

Change the independent variable from x to y in

6. $\frac{d^2y}{dx^2} + (e^y - x)\left(\frac{dy}{dx}\right)^3 = 0$. *Ans.* $\frac{d^2x}{dy^2} + x = e^y$.

7. $\left(3a\frac{dy}{dx} + 2\right)\left(\frac{d^2y}{dx^2}\right)^2 = \left(a\frac{dy}{dx} + 1\right)\frac{dy}{dx}\frac{d^3y}{dx^3}$.
Ans. $\left(\frac{d^2x}{dy^2}\right)^2 = \left(\frac{dx}{dy} + a\right)\frac{d^3x}{dy^3}$.

8. Change the independent variable from x to θ in

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}, \text{ given } x = \rho\cos\theta,\ y = \rho\sin\theta,$$

ρ being a function of θ. *Ans.* The value of R in § 116.

CHAPTER XI.

FUNCTIONS OF TWO OR MORE VARIABLES.

133. Partial differentials and derivatives.

Let $$u = f(x, y),$$

where x and y are both independent. The differential of u as a function of x, y being regarded as constant, is denoted by $\partial_x u$; and the differential of u when y alone is variable is denoted by $\partial_y u$. These differentials are called the *partial differentials* of u with respect to x and y, respectively.

The *partial derivatives* of u with respect to x and y are denoted by $\partial u / dx$ and $\partial u / dy$, respectively.

For example, if $$u = x^2 / a^2 + y^2 / b^2,$$

$$\partial u / dx = 2\,x / a^2, \qquad \partial u / dy = 2\,y / b^2.$$

$$\therefore \frac{x}{2}\frac{\partial u}{dx} + \frac{y}{2}\frac{\partial u}{dy} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = u.$$

EXAMPLES.

1. When $u = by^2x + cx^2 + gy^3 + ex$,

$$\partial_x u = (by^2 + 2\,cx + e)\,dx, \quad \partial_y u = (2\,bxy + 3\,gy^2)\,dy.$$

2. $u = \log x^y, \quad x \cdot \partial u / dx + \partial u / dy = y + \log x.$

3. $u = y^x, \quad \partial u / dx + (y / x) \cdot \partial u / dy = u\,(\log y + 1).$

4. $u = \log (e^x + e^y), \quad \partial u / dx + \partial u / dy = 1.$

5. $u = \log (x + \sqrt{x^2 + y^2}), \quad x \cdot \partial u / dx + y \cdot \partial u / dy = 1.$

6. $u = xy / (x + y), \quad x \cdot \partial u / dx + y \cdot \partial u / dy = u.$

7. $u = x^y y^x, \quad x \cdot \partial u / dx + y \cdot \partial u / dy = (x + y + \log u)\,u.$

134. Total differentials. If we differentiate $u=f(x, y)$, supposing x and y both to vary, we obtain the *total* differential, du, or $df(x, y)$.

The proofs in §§ 16–28 hold equally well when u, y, and z denote functions of two or more independent variables; hence, the total differential of $f(x, y)$ may be obtained by the principles in those articles.

135. *The total differential of a function of two or more variables is equal to the sum of its partial differentials.*

By §§ 16–28 we know that all the terms of $df(x, y)$ are linear in dx and dy. Hence, if $u=f(x, y)$, we may write

$$du=\phi(x, y)\,dx+\phi_1(x, y)\,dy, \tag{1}$$

where $\phi(x, y)$ and $\phi_1(x, y)$ denote, respectively, the sums of the coefficients of dx and dy in the different terms of du.

When x alone varies, (1) becomes

$$\partial_x u=\phi(x, y)\,dx. \tag{2}$$

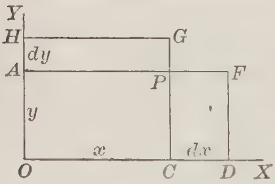
When y alone varies, (1) becomes

$$\partial_y u=\phi_1(x, y)\,dy. \tag{3}$$

From (1), (2), and (3), we obtain

$$du=\partial_x u+\partial_y u.$$

To illustrate this theorem geometrically, let $P(x, y)$ be a moving point in the first quadrant XOY, x and y both being independent.



Let CD and AH, respectively, represent what Δx and Δy would be, if at the values OC and OA the change of each x and y became uniform with respect to the same variable; then $CD=dx$ and $AH=dy$.

Let $\qquad u=$ area of rectangle $OCPA=xy$;

then $\qquad \partial_x u=CDFP=ydx$,

$\qquad \partial_y u=APGH=xdy$,

and $\qquad du=CDFP+APGH$

$\qquad =\partial_x u+\partial_y u$.

Notation. It is often convenient to denote $\partial_x u$ by $\dfrac{\partial u}{dx}\,dx$ or $\dfrac{\partial}{dx}\,u\cdot dx$, and $\partial_y u$ by $\dfrac{\partial u}{dy}\,dy$ or $\dfrac{\partial}{dy}\,u\cdot dy$.

136. *If* $\mathrm{u}=\mathrm{f}(\mathrm{x},\mathrm{y})=\mathrm{a}$, $\quad \dfrac{\mathrm{dy}}{\mathrm{dx}}=-\dfrac{\partial \mathrm{u}/\mathrm{dx}}{\partial \mathrm{u}/\mathrm{dy}}$. (1)

If $u=f(x,y)=a$, by § 135, we have

$$\frac{\partial u}{dx}\,dx+\frac{\partial u}{dy}\,dy=du=0. \tag{2}$$

Solving (2) for dy/dx, we obtain (1).

Note. This formula for the derivative of an implicit function in terms of its partial derivatives is often useful and should be fixed in mind.

Ex. Given $y^3-2x^2y+bx=a=u$, to find dy/dx.

Here $\partial u/dx=-4xy+b$,

and $\partial u/dy=3y^2-2x^2$;

$\therefore dy/dx=(4xy-b)/(3y^2-2x^2)$. by (1)

EXAMPLES.

By § 135 find du when

1. $u=bxy^2+cx^2+gy^3$. $\quad du=(by^2+2cx)\,dx+(2bxy+3gy^2)\,dy$.
2. $u=y^x$. $\quad du=y^x\log y\,dx+xy^{x-1}\,dy$.
3. $u=\log x^y$. $\quad du=x^{-1}y\,dx+\log x\,dy$.
4. $u=\tan^{-1}(y/x)$. $\quad du=(x\,dy-y\,dx)/(x^2+y^2)$.
5. $u=y^{\sin x}$. $\quad du=y^{\sin x}\log y\,\cos x\,dx+y^{\sin x-1}\sin x\,dy$.

By § 136 find dy/dx when

6. $x^3+y^3-3axy=0$. $\quad dy/dx=(x^2-ay)/(ax-y^2)$.
7. $x^m/a^m+y^m/b^m=1$. $\quad dy/dx=-(x/y)^{m-1}(b/a)^m$.
8. $x^y-y^x=0$. $\quad dy/dx=(y^2-xy\log y)/(x^2-xy\log x)$.
9. $x\log y-y\log x=0$. $\quad \dfrac{dy}{dx}=\dfrac{y}{x}\cdot\dfrac{x\log y-y}{y\log x-x}$.

137. If $u = f(x, y, z)$, $y = \phi(x)$, and $z = \phi_1(x)$, u is *directly* a function of x and *indirectly* a function of x through y and z. The differentiation of such functions is often simplified by using the formulas in the next article.

138. *If* $\quad u = f(x, y, z),\ y = \phi(x),$ *and* $z = \phi_1(x)$,

$$\frac{du}{dx} = \frac{\partial u}{dx} + \frac{\partial u}{dy}\frac{dy}{dx} + \frac{\partial u}{dz}\cdot\frac{dz}{dx}, \tag{1}$$

where du/dx *is the total derivative of* u *as a function of* x.

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{dz}dz. \qquad \S\ 135$$

Dividing by dx, we obtain (1).

Cor. 1. If $\quad u = f(y, z),\ y = \phi(x),$ and $z = \phi_1(x)$,

$$\frac{du}{dx} = \frac{\partial u}{dy}\frac{dy}{dx} + \frac{\partial u}{dz}\frac{dz}{dx}. \tag{2}$$

Cor. 2. If $u = f(y)$ and $y = \phi(x)$, $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$.

EXAMPLES.

1. $u = z^2 + y^3 + zy,\ z = \sin x,\ y = e^x$; find du/dx.

Here $\quad \partial u/dy = 3y^2 + z, \quad \partial u/dz = 2z + y,$
$dz/dx = \cos x, \quad dy/dx = e^x.$

Substituting these values in (2) of § 138, we have

$$\begin{aligned} du/dx &= (3y^2 + z)e^x + (2z + y)\cos x \\ &= (3e^{2x} + \sin x)e^x + (2\sin x + e^x)\cos x \\ &= 3e^{3x} + e^x(\sin x + \cos x) + \sin 2x. \end{aligned}$$

2. $u = \tan^{-1}(xy),\ y = e^x.$ $\qquad du/dx = e^x(1 + x)/(1 + x^2e^{2x}).$

3. $u = e^{ax}(y - z),\ y = a\sin x,\ z = \cos x.$ $\quad du/dx = (a^2 + 1)e^{ax}\sin x.$

4. $u = \tan^{-1}\dfrac{y}{x},\ x^2 + y^2 = r^2.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{r^2 - x^2}}.$

5. $u = \sin\dfrac{z}{y},\ z = e^x,\ y = x^2.$ $\qquad \dfrac{du}{dx} = (x - 2)\dfrac{e^x}{x^3}\cos\dfrac{e^x}{x^2}.$

6. $u = \sqrt{x^2 + y^2}$, $y = mx + c$. $\qquad \dfrac{du}{dx} = \dfrac{(1 + m^2)x + mc}{\sqrt{x^2 + (mx + c)^2}}$.

7. $u = \sin^{-1}(y - z)$, $y = 3x$, $z = 4x^3$. $\qquad du/dx = 3/\sqrt{1 - x^2}$.

8. $u = x^4y^2 - x^4y/2 + x^4$, $y = \log x$. $\qquad du/dx = x^3[4(\log x)^2 + 7/2]$.

139. Partial differentials and derivatives of higher orders. If we suppose only one of the independent variables to vary at the same time, by successive differentiations we obtain the *successive partial differentials* $\partial^2{}_x u$, $\partial^2{}_y u$, $\partial^3{}_x u$, $\partial^3{}_y u$, $\cdots$,

$$\text{or} \qquad \frac{\partial^2 u}{dx^2}dx^2, \quad \frac{\partial^2 u}{dy^2}dy^2, \quad \frac{\partial^3 u}{dx^3}dx^3, \quad \frac{\partial^3 u}{dy^3}dy^3, \quad \cdots.$$

For example, if $u = x^2 + xy^2 + y^2$,

$$\partial_x u = (2x + y^2)\,dx, \qquad \partial^2{}_x u = 2\,dx^2, \qquad \partial^3{}_x u = 0;$$
$$\partial_y u = (2xy + 2y)\,dy, \quad \partial^2{}_y u = (2x + 2)\,dy^2, \quad \partial^3{}_y u = 0.$$

If we differentiate u with respect to x, then this result with respect to y, we obtain the *second partial differential*,

$$\partial^2{}_{xy}u, \text{ or } \frac{\partial^2 u}{dxdy}dxdy.$$

For example, if $u = x^3 + x^2y^2$,

$$\partial_x u = (3x^2 + 2xy^2)\,dx, \quad \partial^2{}_{xy}u = 4\,xydxdy.$$

Similarly, the *third* partial differential $\partial^3{}_{yx^2}u$, or $\dfrac{\partial^3 u}{dydx^2}dydx^2$, denotes the result obtained by differentiating u once with respect to y, then this result twice successively with respect to x.

The symbols for the *partial derivatives* are

$$\frac{\partial^2 u}{dx^2}, \quad \frac{\partial^2 u}{dxdy}, \quad \frac{\partial^2 u}{dy^2}, \quad \frac{\partial^3 u}{dx^3}, \quad \frac{\partial^3 u}{dydx^2}, \quad \cdots.$$

In finding the successive partial differentials and derivatives of u, or $f(x, y)$, we treat dx and dy as constants, since x and y are independent variables.

140. If $u = f(x, y)$, $$\frac{\partial^2 u}{dx\,dy} \equiv \frac{\partial^2 u}{dy\,dx}, \tag{1}$$

$$\frac{\partial^3 u}{dx^2 dy} \equiv \frac{\partial^3 u}{dx\,dy\,dx} \equiv \frac{\partial^3 u}{dy\,dx^2}, \text{ etc.}; \tag{2}$$

that is, *if* u *is differentiated successively* m *times with respect to* x *and* n *times with respect to* y, *the result is independent of the order of these differentiations.*

Suppose x alone to vary in $u = f(x, y)$; then

$$\Delta_x u = f(x + \Delta x, y) - f(x, y). \tag{3}$$

Supposing y alone to vary in (3), we obtain

$$\Delta_y(\Delta_x u) = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y).$$

In like manner we find

$$\Delta_x(\Delta_y u) = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y).$$

$$\therefore \Delta_y(\Delta_x u) \equiv \Delta_x(\Delta_y u),$$

or $$\frac{\Delta_y}{\Delta y}\left(\frac{\Delta_x u}{\Delta x}\right) \equiv \frac{\Delta_x}{\Delta x}\left(\frac{\Delta_y u}{\Delta y}\right). \tag{4}$$

Every term in $\Delta_x u / \Delta x$ which contains Δx vanishes in the limit; again, every term in $\frac{\Delta_y}{\Delta y} \operatorname{lt} \frac{\Delta_x u}{\Delta x}$ which contains Δy vanishes in the limit; hence, every term in $\frac{\Delta_y}{\Delta y} \frac{\Delta_x u}{\Delta x}$ which contains either Δx or Δy vanishes in the limit.

Likewise every term in $\frac{\Delta_x}{\Delta x} \frac{\Delta_y u}{\Delta y}$ which contains either Δx or Δy vanishes in the limit.

Hence, by (4) we have

$$\operatorname{lt} \frac{\Delta_y}{\Delta y}\left(\operatorname{lt} \frac{\Delta_x u}{\Delta x}\right) \equiv \operatorname{lt} \frac{\Delta_x}{\Delta x}\left(\operatorname{lt} \frac{\Delta_y u}{\Delta y}\right), \text{ or (1).}$$

Differentiating (1) with respect to x, we obtain

$$\frac{\partial}{dx}\left(\frac{\partial^2 u}{dxdy}\right) \equiv \frac{\partial}{dx}\left(\frac{\partial^2 u}{dydx}\right), \text{ or } \frac{\partial^3 u}{dxdydx} \equiv \frac{\partial^3 u}{dydx^2}. \tag{5}$$

Applying the principle in (1) to $\partial u / dx$, we have

$$\frac{\partial}{dy} \cdot \frac{\partial}{dx}\left(\frac{\partial u}{dx}\right) \equiv \frac{\partial}{dx}\,\frac{\partial}{dy}\left(\frac{\partial u}{dx}\right), \text{ or } \frac{\partial^3 u}{dx^2dy} \equiv \frac{\partial^3 u}{dxdydx}. \tag{6}$$

From (5) and (6), we obtain (2); and so on.

Cor. If, when $\Delta x = i$ and $\Delta y = vi$, we have

$$\Delta^2_{xy} u = \phi(x, y)\,\Delta x \Delta y + vi^n, \text{ where } n > 2;$$

then

$$\partial^2_{xy} u = \phi(x, y)\, dxdy. \tag{7}$$

EXAMPLES.

Verify the identities (1) and (2) of § 140 in each of the four following functions:

1. $u = \cos(x + y)$.
2. $u = x^3y^2 + ay^3$.
3. $u = \tan^{-1}(y/x)$.
4. $u = \sin(bx^5 + ay^5)$.

5. If $u = (x + y)^2$, $x\dfrac{\partial^2 u}{dx^2} + y\dfrac{\partial^2 u}{dxdy} = \dfrac{\partial u}{dx}$.

6. If $u = \dfrac{x^2y^2}{x + y}$, $x\dfrac{\partial^2 u}{dx^2} + y\dfrac{\partial^2 u}{dxdy} = 2\dfrac{\partial u}{dx}$.

7. If $u = (x^2 + y^2)^{1/2}$, $x^2\dfrac{\partial^2 u}{dx^2} + 2xy\dfrac{\partial^2 u}{dxdy} + y^2\dfrac{\partial^2 u}{dy^2} = 0$.

8. If $u = (x^3 + y^3)^{1/2}$, $x^2\dfrac{\partial^2 u}{dx^2} + 2xy\dfrac{\partial^2 u}{dxdy} + y^2\dfrac{\partial^2 u}{dy^2} = \dfrac{3}{4}u$.

9. If $u = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$, $\dfrac{\partial^2 u}{dx^2} + \dfrac{\partial^2 u}{dy^2} + \dfrac{\partial^2 u}{dz^2} = 0$.

10. If $u = e^{xyz}$, $\dfrac{\partial^3 u}{dxdydz} = (1 + 3xyz + x^2y^2z^2)\,u$.

11. If $u = \sin^{-1}(xyz)$, $\dfrac{\partial^3 u}{dxdydz} = \dfrac{1 + 2x^2y^2z^2}{(1 - x^2y^2z^2)^{5/2}}$.

141. *To find the successive total differentials of a function of two independent variables in terms of its partial differentials.*

Let $u = f(x, y)$; then, by § 135, we have

earn to
rite.

$$du = \frac{\partial u}{dx}\,dx + \frac{\partial u}{dy}\,dy. \tag{1}$$

$$\therefore\ d^2u = \frac{\partial^2 u}{dx^2}\,dx^2 + \frac{\partial^2 u}{dxdy}\,dxdy + \frac{\partial^2 u}{dydx}\,dydx + \frac{\partial^2 u}{dy^2}\,dy^2,$$

or

$$d^2u = \frac{\partial^2 u}{dx^2}\,dx^2 + 2\,\frac{\partial^2 u}{dxdy}\,dxdy + \frac{\partial^2 u}{dy^2}\,dy^2. \tag{2}$$

Differentiating (2), remembering that, in general, each term in the second member is a function of both x and y, and applying the principle of § 140, we obtain

$$d^3u = \frac{\partial^3 u}{dx^3}\,dx^3 + 3\,\frac{\partial^3 u}{dx^2dy}\,dx^2dy + 3\,\frac{\partial^3 u}{dxdy^2}\,dxdy^2 + \frac{\partial^3 u}{dy^3}\,dy^3.$$

By successive differentiations, we obtain d^4u, d^5u, etc.

From the analogy between these results and the binomial theorem, the formula for d^nu is easily written out.

142. Expansion of f(x + h, y + k). Regarding x as the only variable, by Taylor's theorem we obtain

$$f(x + h, y + k) = f(x, y + k) + h\,\frac{\partial}{dx}\,f(x, y + k) + \frac{h^2}{\underline{|2}}\,\frac{\partial^2}{dx^2}\,f(x, y + k) + \cdots. \tag{1}$$

Regarding y as the only variable, we obtain

$$f(x, y + k) = f(x, y) + \frac{k}{1}\,\frac{\partial}{dy}\,f(x, y) + \frac{k^2}{\underline{|2}}\,\frac{\partial^2}{dy^2}\,f(x, y) + \cdots.$$

Substituting in (1) this value of $f(x, y + k)$, we obtain

$$f(x + h, y + k) = f(x, y) + h\,\frac{\partial}{dx}\,f(x, y) + k\,\frac{\partial}{dy}\,f(x, y)$$
$$+ \frac{1}{\underline{|2}}\left[h^2\,\frac{\partial^2}{dx^2}\,f(x,y) + 2\,hk\,\frac{\partial^2}{dxdy}\,f(x,y) + k^2\,\frac{\partial^2}{dy^2}\,f(x,y)\right] + \cdots. \tag{C}$$

A symbolic expression for this formula is

$$f(x+h,\,y+k)=f(x,\,y)+\left(h\frac{\partial}{dx}+k\frac{\partial}{dy}\right)f(x,\,y)$$
$$+\frac{1}{\lfloor 2}\left(h\frac{\partial}{dx}+k\frac{\partial}{dy}\right)^2 f(x,\,y)$$
$$+\frac{1}{\lfloor 3}\left(h\frac{\partial}{dx}+k\frac{\partial}{dy}\right)^3 f(x,\,y)+\cdots, \qquad (C')$$

where

$$\left(h\frac{\partial}{dx}+k\frac{\partial}{dy}\right)^n$$

is to be expanded by the binomial theorem and $f(x,\,y)$ written after each of the resulting terms.

If $u=f(x,\,y)$, we may put u for $f(x,\,y)$ in (C) and (C′).

Compare (C) with (1) in § 144.

Cor. 1. By a similar course of reasoning Taylor's theorem is extended to the expansion of functions of three or more independent variables, in series analogous to that in (C), or (C′).

Cor. 2. By Taylor's theorem we obtain the value of $f(x+h)-f(x)$ in ascending powers of h; that is, Taylor's theorem expresses the increment of $f(x)$ in ascending powers of the increment of x. Similarly, by the extension of Taylor's theorem, we express the increment of a function of two or more independent variables in ascending powers of the increments of those variables.

143. Maxima and minima of f(x, y).

$f(a,\,b)$ is a *maximum* of $f(x,\,y)$ when, for all small positive or negative values of h and k,

$$f(a+h,\,b+k)-f(a,\,b)<0.$$

$f(a,\,b)$ is a *minimum* of $f(x,\,y)$ when, for all small positive or negative values of h and k,

$$f(a+h,\,b+k)-f(a,\,b)>0.$$

144. Conditions for maxima and minima of f(x, y). Putting $u = f(x, y)$, from (C) of § 142, we obtain

$$f(x+h, y+k) - f(x, y) = h\frac{\partial u}{dx} + k\frac{\partial u}{dy}$$
$$+ \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 u}{dx^2} + 2\,hk\frac{\partial^2 u}{dxdy} + k^2\frac{\partial^2 u}{dy^2}\right) + \cdots. \quad (1)$$

When h and k are very small, the quality of

$$f(x+h, y+k) - f(x, y),$$

or of the second member of (1), will evidently depend upon the quality of h and k unless

$$\partial u / dx = 0, \quad \text{and} \quad \partial u / dy = 0. \qquad \text{(a)}$$

But, by definition, when $f(x, y)$ reaches a maximum or a minimum value, the quality of $f(x+h, y+k) - f(x, y)$ is independent of h and k. Hence, equations (a) express one condition for a maximum or a minimum of $f(x, y)$.

Suppose $x = a$, $y = b$ to be one solution of system (a).

Let A, B, C denote the values of

$$\partial^2 u / dx^2, \ \partial^2 u / dxdy, \ \partial^2 u / dy^2,$$

respectively, when $x = a$ and $y = b$; then from (1), we have

$$f(a+h, b+k) - f(a, b) = \frac{1}{\underline{|2}}(Ah^2 + 2\,Bhk + Ck^2) + \cdots$$
$$= \frac{(Ah + Bk)^2 + (AC - B^2)\,k^2}{A\underline{|2}} + \cdots. \qquad (2)$$

The quality of the second member of (2) is independent of h and k when and only when $AC - B^2$ is positive or zero.* For when $AC - B^2$ is negative, the numerator will be positive when $k = 0$, and negative when $Ah + Bk = 0$. Hence, a second condition for a maximum or a minimum of $f(x, y)$ is

$$AC > B^2, \quad \text{or} \quad \frac{\partial^2 u}{dx^2}\cdot\frac{\partial^2 u}{dy^2} > \left(\frac{\partial^2 u}{dxdy}\right)^2. \qquad \text{(b)}$$

---

* The limits of this treatise exclude the investigation of the exceptional case when $AC = B^2$ or when $A = B = C = 0$.

When condition (b) is satisfied by $x = a$, $y = b$; from (2) we see that $f(a, b)$ will be a maximum or a minimum of $f(x, y)$ according as A, or $\partial^2 u / dx^2]_{a, b}$, is negative or positive.

Cor. 1. Condition (b) requires that $\partial^2 u / dx^2]_{a, b}$ and $\partial^2 u / dy^2]_{a, b}$ have the same quality.

Note. This discussion assumes that $f(x, y)$ and all its successive derivatives are continuous functions.

Cor. 2. By a similar course of reasoning we may obtain the conditions for maxima and minima of functions of three or more variables.

EXAMPLES.

Examine for maxima and minima

1. $u = f(x, y) = x^2 y + xy^2 - axy$.

$$\partial u / dx = (2x + y - a)y, \qquad \partial u / dy = (2y + x - a)x,$$
$$\partial^2 u / dx^2 = 2y, \qquad \partial^2 u / dy^2 = 2x,$$
$$\partial^2 u / dxdy = 2x + 2y - a.$$

Hence, condition (a) of § 144 is

$$(2x + y - a)y = 0, \quad (2y + x - a)x = 0; \tag{1}$$

and condition (b) is

$$4xy > (2x + 2y - a)^2. \tag{2}$$

System (1) has the four solutions $(0, 0)$, $(a, 0)$, $(0, a)$, $(a/3, a/3)$. The last, and it only, satisfies (2); hence, $f(a/3, a/3)$ is a maximum or a minimum of $f(x, y)$.

If a is positive, $\partial^2 u / dx^2$ is positive when $y = a/3$; hence,

$$f(a/3, a/3), \quad \text{or} \quad -a^3/27,$$

is a minimum. If a is negative, $\partial^2 u / dx^2$ is negative when $y = a/3$; hence, $-a^3/27$ is a maximum.

2. $u = x^3 + y^3 - 3axy$.

Ans. When a is $+$, $-a^3$ is a min.; when a is $-$, $-a^3$ is a max.

3. $u = x^2 + xy + y^2 - ax - by$. *Ans.* $(ab - a^2 - b^2)/3$ is a min.

4. $u = x^3 y^2 (a - x - y)$. *Ans.* $a^6/432$ is a max.

5. $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. *Ans.* -8 is a min.

6. $u = (2ax - x^2)(2by - y^2)$. *Ans.* a^2b^2 is a max.

7. Find the maximum of xyz subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Since xyz is arithmetically a maximum when $x^2y^2 \cdot z^2/c^2$ is a maximum, we put

$$u = x^2y^2\frac{z^2}{c^2} = x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Here condition (a) of § 144 is

$$2xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0, \quad 2x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0; \qquad (1)$$

and condition (b) is

$$\left(1 - \frac{6x^2}{a^2} - \frac{y^2}{b^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{6y^2}{b^2}\right) > 4\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2}\right)^2. \qquad (2)$$

The only solution of system (1) which satisfies (2) is

$$x = a/\sqrt{3}, \ y = b/\sqrt{3}.$$

$$\frac{\partial^2 u}{dx^2} = 2y^2\left(1 - \frac{6x^2}{a^2} - \frac{y^2}{b^2}\right) = -\frac{8b^2}{9}, \text{ when } x = \frac{a}{\sqrt{3}}, \ y = \frac{b}{\sqrt{3}}.$$

Hence, $abc/3\sqrt{3}$ is a maximum of xyz.

8. Given the sum of the three edges of a rectangular parallelopiped; find its form when its volume is a maximum. *Ans.* A cube.

9. Divide m into three parts x, y, z such that $x^a y^b z^c$ may be a maximum. *Ans.* $x/a = y/b = z/c = m/(a + b + c)$.

10. Divide 24 into three such parts that the continued product of the first, the square of the second, and the cube of the third may be a maximum. *Ans.* 4, 8, 12.

11. The vertices of a triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3); find the point the sum of the squares of whose distances from the vertices is a minimum.

Ans. $[(x_1 + x_2 + x_3)/3, \ (y_1 + y_2 + y_3)/3]$, or the centre of gravity of the triangle.

CHAPTER XII.

ASYMPTOTES. SINGULAR POINTS. CURVE TRACING.

145. An **asymptote** to a curve is a fixed line which is the limit of a tangent when the point of contact moves out along an infinite branch of the curve.

146. *To obtain the equations of the asymptotes to the curve*

$$f(x, y) = 0, \tag{1}$$

where f (x, y) *is of the* n*th degree in* x *and* y.

Let

$$y = mx + l \tag{2}$$

be the equation of a tangent to (1).

Substituting $mx + l$ for y in (1), we obtain

$$f(x, mx + l) = 0. \tag{3}$$

Equations (2) and (3) form a system which is equivalent to the system (1) and (2); hence, the n roots of (3) are the abscissas of the n points common to the curve (1) and its tangent (2).

Since (2) is a tangent to (1), two roots of (3) are equal.

Conceive the point of contact to move out along an infinite branch; then the two equal roots of (3) will become infinites.

Therefore, by algebra,* the coefficients of x^n and x^{n-1} in (3) will approach zero as their common limit.

---

* Substituting $1/x$ for x in (1), and multiplying by x^n, we obtain (2).

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0. \tag{1}$$

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + 1 = 0. \tag{2}$$

The n roots of (2) are the reciprocals of the n roots of (1).

When $p_n \doteq 0$ and $p_{n-1} \doteq 0$, two roots of (1) $\doteq 0$; $\therefore$ two roots of (2) $= \infty$.

When $p_n = 0$ and $p_{n-1} = 0$, two roots of (1) $= 0$; $\therefore$ two of the n roots of (2) assume the form $a\rho$, and (2) has, in reality, only $n - 2$ roots.

Hence, by putting the coefficients of x^n and x^{n-1} equal to zero, and solving the resulting system of equations for m and l, we obtain the slope and the intercept of each asymptote.

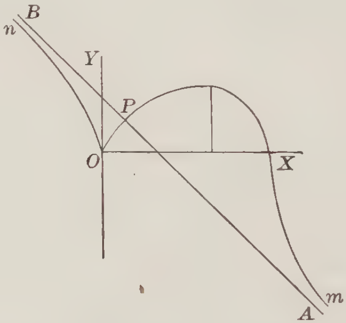
Ex. Examine for asymptotes the curve,

$$y^3 = ax^2 - x^3. \qquad (1')$$

Let
$$y = mx + l \qquad (2')$$
be the equation of an asymptote to (1′).

Substituting $mx + l$ for y in (1′) and arranging the terms according to the powers of x, we have

$$(m^3+1)x^3 + (3m^2l - a)x^2 + 3ml^2x + l^3 = 0.$$



Putting the coefficients of x^3 and x^2 equal to zero, we have

$$m^3 + 1 = 0, \qquad 3m^2l - a = 0. \qquad (3')$$

The only real solution of system (3′) is $m = -1$, $l = a/3$.

Substituting these values in (2′), we obtain

$$y = -x + a/3, \qquad (4')$$

which is the only real asymptote to curve (1′).

The locus of (1′) is the curve $nOPm$, and that of (4′) is the asymptote AB.

Cor. 1. If we expand $f(x, mx + l)$ and arrange the result according to the descending powers of x, (3) will assume the form

$$A_nx^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \cdots + A_1x + A_0 = 0. \qquad (4)$$

A_n will be of the nth degree in m, but will not contain l.

A_{n-1} will be linear in l.

Hence, the system, $A_n = 0$ and $A_{n-1} = 0$, (when determinate) has at most only n solutions; whence *a curve of the* n*th order cannot, in general, have more than* n *asymptotes.*

When, as is assumed in this article, equation (1) is of the nth degree in y, there will be no asymptote parallel to the y-axis; and conversely (§ 148).

Cor. 2. The n roots of equation (4) are the abscissas of the n points common to the curve (1) and the line (2).

When $A_n = 0$ and $A_{n-1} = 0$, (4) is, in reality, of the $(n-2)$th degree and has only $n-2$ roots, the other two of the n roots assume the form ∞.

Hence, *an asymptote to the curve* (1) *cannot have more than* n − 2 *points in common with the curve.*

For example, (1′) and (4′) form a system which is defective in two solutions; hence, the asymptote AB, or (4′), has only the one point P in common with the curve $nOPm$, or (1′).

Any line parallel to an asymptote has a slope m which satisfies $A_n = 0$; but when $A_n = 0$, (4) has only $n - 1$ roots.

Hence, *any line parallel to an asymptote to the curve* (1) *cannot have more than* n − 1 *points in common with the curve.*

For example, (1′) and $y = -x + c$ form a system which is defective in one solution; hence, any line parallel to AB cannot have more than two points in common with the curve $nOPm$.

EXAMPLES.

Examine for asymptotes

1. The folium of Descartes $x^3 + y^3 = 3axy$. § 155, fig. 6
2. $y^3 = ax^2 + x^3$. $y = x + a/3$.
3. The conic sections. $ay = \pm bx$.
4. $(y^2 - 1)y = (x^2 - 4)x$. $y = x$.
5. $y^4 - x^4 + 2ax^2y = b^2x^2$. $y = \pm x - a/2$.

6. $x^4 - y^4 - a^2xy = b^2y^2$. $y = \pm x$.

7. $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$. $y = x,\ y = 2x, y = 3x$.

8. $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$.
$y = -x/3 - 3/4,\ y = x + 1/4,\ y = -x + 3/2$.

9. Prove that the asymptote $y = -x$ lacks three points of intersection with its curve $x^3 + y^3 = a^3$.

147. Parallel asymptotes. In (4) of § 146 it sometimes happens that $A_{n-1} = 0$ does not determine l for one or more of the values of m given by $A_n = 0$. If, in this case, l is determined by $A_{n-2} = 0$, which is a quadratic equation in l, we shall obtain for each value of m two values for l. This gives *two parallel* asymptotes, each of which lacks three points of intersection with its curve; for, in this case, (4) in § 146 has only $n - 3$ roots, the other three roots assume the form ∞.

If l is not determined by $A_{n-2} = 0$, but is determined by $A_{n-3} = 0$, we obtain *three parallel* asymptotes; and so on.

COR. Any one of k parallel asymptotes lacks $k + 1$ points of intersection with its curve; and any line parallel to k parallel asymptotes lacks k points of intersection with its curve.

Ex. Examine $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$ for asymptotes.

Substituting $mx + l$ for y, we obtain

$$(m^3 - m^2 - m + 1)x^3 + (3m^2l - m^2 - 2ml - l + 1)x^2 + (3ml^2 - 2ml - l^2)x + (l^3 - l^2 - 1) = 0.$$

Equating the coefficients of x^3 and x^2 to zero, we have

$$\left.\begin{aligned} m^3 - m^2 - m + 1 &= 0, &(1)\\ 3m^2l - m^2 - 2ml - l + 1 &= 0. &(2)\end{aligned}\right\}$$

From (1), $m = -1$ or 1.

When $m = -1$, from (2) we obtain $l = 0$; hence, one asymptote is

$$y = -x.$$

When $m = 1$, (2) does not determine l. This indicates that there are *parallel* asymptotes having the slope 1. To obtain the values of l for these asymptotes, we equate the coefficient of x to zero, and obtain

$$3\,ml^2 - 2\,ml - l^2 = 0. \qquad (3)$$

When $m = 1$, from (3) we obtain $l = 0$ or 1; hence, the parallel asymptotes are $y = x$ and $y = x + 1$.

148. Asymptotes parallel to either axis. In § 146, let the axes be revolved until the y-axis is parallel to an asymptote; then one value of m will assume the form ∞; hence, $A_n = 0$ will be below the nth degree in m, and therefore $f(x, y) = 0$ will be below the nth degree in y.

Conversely, if $f(x, y) = 0$ is below the nth degree in y, there will be one or more asymptotes parallel to the y-axis.

Hence, if $f(x, y) = 0$ is below the nth degree in either x or y, there will be one or more asymptotes parallel to a co-ordinate axis. To find the equations of these asymptotes, *equate to zero the coefficients of the highest powers of* x *and* y.

The following example will make clear this principle:

Ex. Find the asymptotes of the curve,

$$y^2x^2 - 3\,yx^2 - 5\,xy^2 + 2\,x^2 + 6\,y^2 + x + y + 1 = 0. \qquad (1)$$

Arranging (1) in descending powers of x, we have

$$(y^2 - 3\,y + 2)\,x^2 - (5\,y^2 - 1)\,x + 6\,y^2 + y + 1 = 0. \qquad (2)$$

Equating the coefficient of x^2 to zero, we obtain

$$y^2 - 3\,y + 2 = 0;\ \text{that is, } y = 1,\ y = 2.$$

Substituting 1 for y, (2) becomes $-x + 2 = 0$.

Hence, (2) and $y = 1$ form a system which is defective in three solutions; that is, (2) and $y = 1$ have only one common point.

Substituting 2 for y, (2) becomes $-19\,x + 27 = 0$; hence, (2) and $y = 2$ form a system which is defective in three solutions.

Therefore, $y = 1$ and $y = 2$ are two parallel asymptotes, which are parallel to the x-axis.

Arranging (1) in descending powers of y, we have

$$(x^2 - 5\,x + 6)\,y^2 - (3\,x^2 - 1)\,y + 2\,x^2 + x + 1 = 0. \qquad (3)$$

From (3) we see that $x = 2$ and $x = 3$ are two parallel asymptotes, which are parallel to the y-axis.

EXAMPLES.

Find the asymptotes to

1. The cissoid of Diocles $(2a - x)y^2 = x^3$.
   See example 3 and fig. 3, § 155.

2. The strophoid $x(x^2 + y^2) + a(x^2 - y^2) = 0$.
   See example 7 and fig. 7, § 155.

3. $xy^3 + x^3y = a^4$. $\quad x = 0,\ y = 0.$

4. $y^2(x^2 - a^2) = x$.

5. $xy - cy - bx = 0$.

6. $x^2y^2 = c^2(x^2 + y^2)$.

7. $(x - b)^2(y - c) = a^3$.

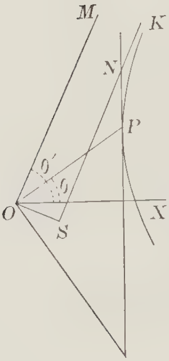
8. $(x - a)y^2 = x^3 + ax^2$. $\quad x = a,\ y = \pm x \pm a.$

9. $x^3 + 2x^2y + xy^2 - x^2 - xy = -2$. $\quad x = 0,\ y = -x,\ y = -x + 1.$

10. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$. $\quad y = \pm x,\ y = x + 1.$

11. $x^2y^2 - x^2y - xy^2 + 2x + 3y + 1 = 0$.
    $y = 0,\ y = 1,\ x = 0,\ x = 1.$

12. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x = 1$.
    $y = x,\ y = 2x + 1,\ y = 2x + 2.$



149. Asymptotes to polar curves. When OP, or ρ, revolves about O, it will become ∞ when, and only when, it is parallel to an asymptote, as SN. Hence, any value of θ which makes $\rho = \infty$ gives the direction of an asymptote, and the corresponding value of the subtangent OS gives the distance and direction of this asymptote from the pole O.

EXAMPLES.

Examine for asymptotes

1. The hyperbolic spiral $\rho\theta = a$.

When $\theta = 0$, $\rho = \infty$, and the subt. $= -a$.

Therefore, the line parallel to the polar axis and at the distance a above it is an asymptote to the spiral.

2. The curve $\rho \cos\theta = a \cos 2\theta$.

When $\theta = \pi/2$, $\rho = \infty$, and subt. $= -a$.

Hence, the line perpendicular to the polar axis at the distance a to the left of the pole is an asymptote.

3. The curve $\rho^2 \cos\theta = a^2 \sin 3\theta$.

4. The curve $\rho = a \sec 2\theta$.

There are four asymptotes forming the sides of a square, each being at the distance $a/2$ from the pole.

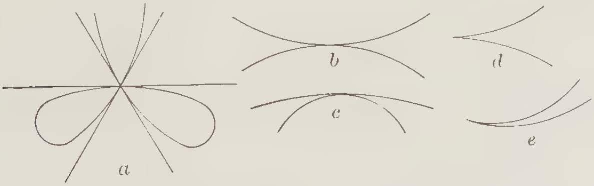
5. The polar axis is an asymptote to the lituus $\rho\sqrt{\theta} = a$.

Singular Points.

150. The **singular points** of a curve are those points which have some peculiar property independent of the position of the co-ordinate axes. *Points of inflexion* and *multiple points* are varieties of singular points.

Points of inflexion have already been considered in § 109.

151. A **multiple point** is one through which two or more branches of a curve pass, or at which they meet.



A multiple point is double when there are only two branches, triple when only three, and so on.

A **multiple point of intersection** is a multiple point at which the branches cross each other (fig. *a*).

An **osculating point** is a multiple point through which the branches pass and at which they are tangent to each other (figs. *b* and *c*).

A **cusp** is a multiple point at which the branches terminate and are tangent to each other (figs. *d* and *e*).

A cusp or osculating point is of the first or the second species according as the two branches are on opposite sides (figs. *b* and *d*) or the same side (figs. *c* and *e*) of their common tangent.

152. *To find the multiple points of a curve.*

At a multiple point each branch has its own tangent; hence, at such a point dy/dx has two or more values.

Let $f(x, y) = 0$ be the algebraic equation of the curve free from radicals and fractions; then, by § 136, we have

$$\frac{dy}{dx} = -\frac{\partial u/dx}{\partial u/dy}, \text{ where } u = f(x, y).$$

Since u contains neither radicals nor fractions, this expression for dy/dx can have only one value at any given point unless it assumes the indeterminate form $0/0$; hence, at a multiple point we have

$$\partial u/dx = 0, \qquad \partial u/dy = 0. \tag{1}$$

Any solution of system (1) which satisfies the equation of the curve gives a point on the curve at which

$$dy/dx = 0/0. \tag{2}$$

The form in (2) can be evaluated by the method of § 88.

If, in (2), dy/dx has two or more unequal real values, the point is, in general, a *multiple point of intersection.*

If, in (2), dy/dx has two equal real values, the point is, in general, an *osculating* point or a cusp.

Ex. 1. Examine the curve $x^4 + ax^2y - ay^3 = 0$ for multiple points.

Here $$u = x^4 + ax^2y - ay^3 = 0; \tag{1}$$

$$\therefore \partial u/dx = 4x^3 + 2axy, \qquad \partial u/dy = ax^2 - 3ay^2. \tag{2}$$

Equating these partial derivatives to zero, we have

$$x(2x^2 + ay) = 0, \quad x^2 - 3y^2 = 0. \tag{3}$$

The only solution of system (3) which will satisfy (1) is $x = 0,\ y = 0$; hence, the only point to be examined is (0, 0).

From (2) and § 136, we have

$$\frac{dy}{dx} = \frac{4x^3 + 2axy}{3ay^2 - ax^2} = \frac{0}{0} \text{ when } \begin{cases} x = 0, \\ y = 0. \end{cases}$$

Evaluating this fraction by the method of § 88, we find

$$dy/dx]_{0,0} = 0,\ +1,\ -1.$$

Hence, the origin is a triple point of intersection at which the inclinations of the branches are, respectively, 0, $\pi/4$, $3\pi/4$.

The general form of the curve at the origin is shown in § 151, fig. *a*.

Ex. 2. Examine the curve $a^4y^2 = a^2x^4 - x^6$ for multiple points.

Here
$$u = a^4y^2 - a^2x^4 + x^6 = 0; \tag{1}$$
$$\therefore \partial u/dx = -4a^2x^3 + 6x^5, \qquad \partial u/dy = 2a^4y. \tag{2}$$

The only solution of the system,

$$-4a^2x^3 + 6x^5 = 0, \qquad 2a^4y = 0, \tag{3}$$

which will satisfy (1) is $x = 0,\ y = 0$.

$$\frac{dy}{dx} = \frac{4a^2x^3 - 6x^5}{2a^4y} = \frac{0}{0} \text{ when } \begin{cases} x = 0, \\ y = 0. \end{cases}$$

Evaluating this fraction, we have

$$dy/dx]_{0,0} = \pm 0.$$

An inspection of its equation shows that the curve passes through the origin and is symmetrical with respect to the x-axis; hence, the origin is an osculating point of the first species (§ 155, fig. 5).

Ex. 3. Examine $y^2 = x(x + a)^2$ for multiple points.

Here
$$\frac{dy}{dx} = \frac{3x^2 + 4ax + a^2}{2y} = \frac{0}{0} \text{ when } \begin{cases} x = -a, \\ y = 0. \end{cases}$$

Evaluating this fraction, we obtain

$$dy/dx]_{-a,0} = \pm\sqrt{-a}.$$

When a is negative, $(-a,\ 0)$ is a double point of intersection.

When a is positive, the slopes of both the branches which pass through the point $(-a,\ 0)$ are imaginary; hence, these branches do not lie in the plane of the axes, but are a part of the imaginary locus.

When a is positive, the multiple point $(-a,\ 0)$ is isolated from the rest of the plane locus, and is called a *conjugate point.*

153. **A conjugate point** is a multiple point which is formed by imaginary branches meeting or crossing each other in the plane of the axes.

Such a point is, in general, entirely isolated from the plane locus; but in exceptional cases it may lie on it.

At a conjugate point dy/dx is, in general, imaginary; but in exceptional cases it may be real, since the tangents to the imaginary branches at such a point may lie in the plane of the axes.

A **shooting point** is a multiple point at which two or more branches end but are not tangent to each other. A **stop point** is a point at which a single branch of a curve ends. As a shooting point or a stop point never occurs on an algebraic curve, they will not be further considered.

EXAMPLES.

Show that the curve

1. $ay^2 = x^3$ has a cusp of the first species at (0, 0). § 155, fig. 2.

2. $y^3 = 2ax^2 - x^3$ has a cusp of the first species at (0, 0). § 155, fig. 4.

3. $x^3 + y^3 = 3axy$ has a double point of intersection at (0, 0). § 155, fig. 6.

4. $y^2 = x^2(a^2 - x^2)$ has a double point of intersection at (0, 0).

The form of this curve is similar to that of the curve in § 155, fig. 13.

5. $x(x^2 + y^2) = a(y^2 - x^2)$ has a double point of intersection at (0, 0). § 155, fig. 7.

6. $y^2(a^2 - x^2) = x^4$ has a point of osculation of the first species at (0, 0).

7. $y^2 = 2x^2y + x^4y - 2x^4$ has a conjugate point at (0, 0).

The slope of each branch at (0, 0) is real; but (0, 0) is an isolated point; hence, it must be a conjugate point at which the tangents to the imaginary branches lie in the plane of the axes.

8. $ay^2 = (x - a)^2(x - b)$ has at $(a, 0)$ a conjugate point when $a < b$, a double point of intersection when $a > b$, and a cusp when $a = b$.

9. $a^3y^2 - 2abx^2y = x^5$ has at (0, 0) a point of osculation and a point of inflexion on one branch.

Curve Tracing.

154. Symmetry. The following principles of symmetry of loci are easily proved:

A locus is symmetrical with respect to the x-axis when its equation contains only even powers of y.

A locus is symmetrical with respect to the y-axis when its equation contains only even powers of x.

A locus is symmetrical with respect to the origin when the terms in its equation are all of an odd or all of an even degree in x and y.

155. To trace the locus of an equation we first find its

Axis or centre of symmetry, if any;
Intercepts on the axes, and its limits;
Maxima and minima ordinates, if any;
Asymptotes and singular points, if any.

It is useful to remember also that an infinite branch is convex toward its asymptote.

EXAMPLES.

1. Trace the cubical parabola $a^2y = x^3$.

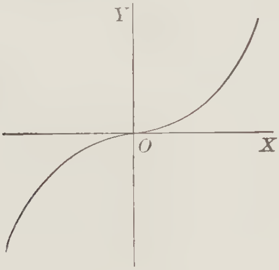


FIG. 1.

The origin is a centre of symmetry and a point of inflexion, to the right of which the curve is concave upward. The infinite branches are in the first and the third quadrant. When $x = \pm\infty$, $\phi = \pi/2$.

For the form of the curve, see fig. 1.

2. Trace the semi-cubical parabola $a^{1/2}y = x^{3/2}$, or $ay^2 = x^3$.

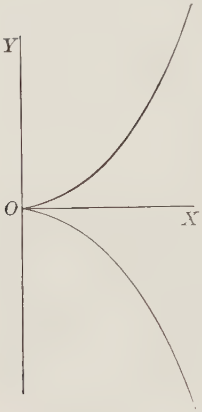


FIG. 2.

The curve is symmetrical with respect to the x-axis, and lies to the right of the y-axis.

The origin is a cusp of the first species.

When $x = \infty$, $\phi \doteq \pi/2$.

For the form of the curve, see fig. 2.

The curve

$$a^{n-1}y = x^n \tag{1}$$

is frequently called the *parabola of the* n*th degree*, n being greater than unity.

When $n = 2$, (1) becomes

$$ay = x^2, \tag{2}$$

the locus of which is the common parabola with its axis on the y-axis.

When n is an even integer or a fraction having an even numerator and an odd denominator, the general form of curve (1) is that of the common parabola (2).

When n is an odd integer or a fraction with an odd numerator and an odd denominator, the general form of curve (1) is that in fig. 1.

When n is a fraction having an odd numerator and an even denominator, the general form of (1) is that in fig. 2.

3. Trace the cissoid of Diocles,

$$y^2 = x^3/(2a - x).$$

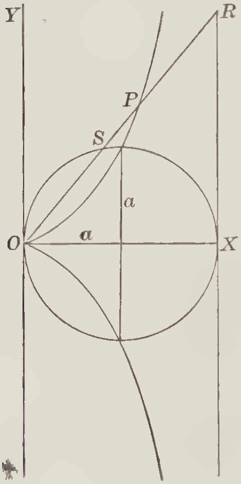


FIG. 3.

The x-axis is an axis of symmetry. The curve passes through (a, a) and $(a, -a)$. It lies between $x = 0$ and the asymptote $x = 2a$.

The origin is a cusp of the first species. (See fig. 3.)

To construct the cissoid geometrically, draw any line OR from O to XR, and take

$$RP = OS;$$

then P will be a point on the cissoid.

4. Trace the curve $y^3 = 2ax^2 - x^3$.

$$y = -x + 2a/3$$

is an asymptote.

$(2a, 0)$ is a point of inflexion, to the right of which the curve is concave upward. Hence, the infinite branch in the fourth quadrant lies above the asymptote, and the one in the second below it.

When $x = 4a/3$,

$$y = 2a\sqrt[3]{4}/3,$$

a maximum ordinate.

The origin is a cusp of the first species, the y-axis being tangent to both branches. (See fig. 4.)

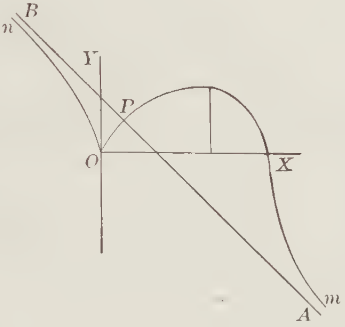


FIG. 4.

5. Trace the curve $a^4y^2 = a^2x^4 - x^6$.

Each axis is an axis of symmetry. The origin is an osculating point of the first species.

The curve is enclosed by the tangents

$x = \pm a$, $y = \pm 2a\sqrt{3}/9$.

When $x = \pm a\sqrt{6}/3$,

$y = 2a\sqrt{3}/9$, a maximum.

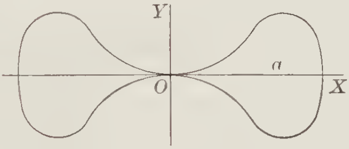


FIG. 5.

When $x = \pm a\sqrt{27 - 3\sqrt{33}}/6$, (x, y) is a point of inflexion. (See fig. 5.)

6. Trace the folium of Descartes $y^3 - 3axy + x^3 = 0$.

$y = -x - a$ is an asymptote.

The infinite branches are concave upward, and hence lie above this asymptote.

$dy/dx = 0$ at (0, 0)

and $(a\sqrt[3]{2},\ a\sqrt[3]{4})$;

$dy/dx = ap$ at (0, 0)

and $(a\sqrt[3]{4},\ a\sqrt[3]{2})$.

This indicates a double point of intersection at the origin and a loop tangent to the axes and the lines

$y = a\sqrt[3]{4}$ and $x = a\sqrt[3]{4}$. (See fig. 6.)

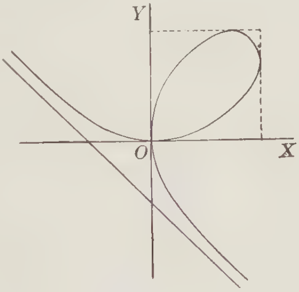


FIG. 6.

7. Trace the strophoid $x(x^2+y^2)+a(x^2-y^2)=0$. (See fig. 7.)

8. Trace the hypocycloid of four cusps $x^{2/3}+y^{2/3}=a^{2/3}$.

Each axis is an axis of symmetry.

The limits of the locus are $x=\pm a$, $y=\pm a$.

$$dy/dx=-(y/x)^{1/3}=0, \text{ at } (-a, 0) \text{ or } (a, 0),$$
$$=\infty, \text{ at } (0, -a) \text{ or } (0, a);$$

hence, there is a cusp of the first species at each of these four points.

$$d^2y/dx^2=a^{2/3}/3x^{4/3}y^{1/3};$$

hence, the curve is concave upward in the first and second quadrants. (See fig. 8.)

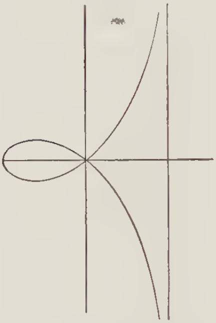


FIG. 7.

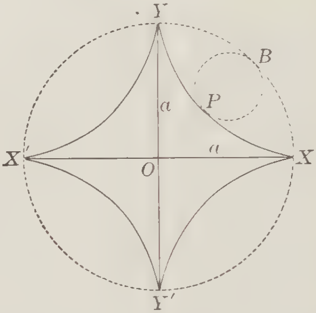


FIG. 8.

In order to examine this curve for cusps by the method of § 152, it would be necessary first to rationalize its equation.

This curve is traced by a point, P, in the circle PB (fig. 8) as it rolls within the fixed circle $XBYX'$, whose radius is four times that of the circle PB.

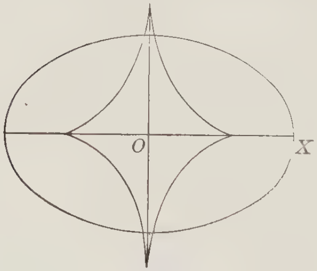


FIG. 9.

9. Trace the evolute of the ellipse,

$$(ax)^{2/3}+(by)^{2/3}=(a^2e^2)^{2/3},$$

where a and b denote the semi-axes, and e the excentricity of the ellipse.

Each axis is an axis of symmetry. The curve is enclosed by the lines $x=\pm ae^2$, $y=\pm a^2e^2/b$.

$$dy/dx=-(a^2y/b^2x)^{1/3}=0,$$

at $(-ae^2, 0)$ or $(ae^2, 0)$;

$$dy/dx=\infty,$$

at $(0, -a^2e^2/b)$ or $(0, a^2e^2/b)$.

Hence, there is a cusp of the first species at each of these four points.

The curve is concave upward in the first and second quadrants. (See fig. 9.)

10. Trace the conchoid $x^2y^2 = (b^2 - y^2)(a + y)^2$.

11. Trace the witch of Agnesi $(x^2 + 4a^2)y = 8a^3$.

12. Trace the curve $x^3 + y^3 = a^3$.

13. Trace the polar curve $\rho = a\cos\theta + b$.

$$\begin{array}{llllll} \text{When} & \theta = 0, & \pi/4, & \pi/2, & \cos^{-1}(-b/a), \\ & \rho = a + b, & \sqrt{2}\,a/2 + b, & b, & 0; \\ \text{when} & \theta = 3\pi/4, & \pi, & \cdots, & \\ & \rho = -\sqrt{2}\,a/2 + b, & -a + b, & \cdots. & \end{array}$$

These values of ρ will recur in reverse order as θ increases from π to 2π; hence, the locus is symmetrical with respect to the polar axis.

Locating these points for $a = 3$ and $b = 2$, we obtain the curve in fig. 10.

When $a > b$, the point O is a double point of intersection, as in the figure.

When $a = b$, O is a cusp.

When $a < b$, there is a point of inflexion at R and at S, and O is not on the curve.

When $a = b$, this curve is called the *cardioid.*

When $a = 2b$, it is called the *limaçon.*

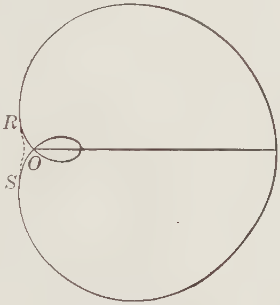


FIG. 10.

14. Trace the curve $\rho = a\sin 3\theta$.

$$\rho = a, \text{ a maximum, when } \sin 3\theta = 1;$$

that is, when $\theta = \pi/6,\ 5\pi/6,\ \cdots,$

$$\rho = -a, \text{ a minimum, when } \sin 3\theta = -1;$$

that is, when $\theta = \pi/2,\ 7\pi/6,\ \cdots.$

$$\begin{array}{lllllllll} \text{When } \theta = 0, & \pi/6, & \pi/3, & \pi/2, & 2\pi/3, & 5\pi/6, & \pi, & \cdots, \\ \rho = 0, & a, & 0, & -a, & 0, & a, & 0, & \cdots. \end{array}$$

The curve consists of *three* equal loops. (See fig. 11.)

15. Trace the curve $\rho = a \sin 2\theta$.

The curve consists of *four* equal loops. (See fig. 12.)

The locus of $\rho = a \sin \theta$ is a circle, a curve of *one* loop.

From the number of loops when $n = 1, 2, 3$, we infer that the locus of $\rho = a \sin n\theta$ consists of n loops when n is odd, and $2n$ loops when n is even.

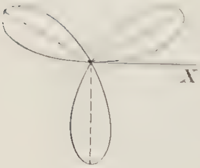


FIG. 11.

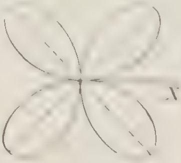


FIG. 12.

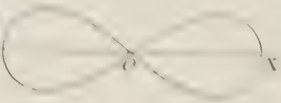


FIG. 13.

16. Trace the lemniscate $\rho^2 = a^2 \cos 2\theta$.

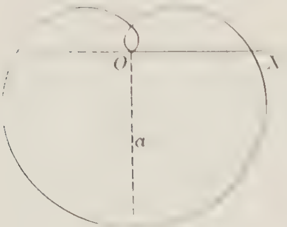


FIG. 14.

The *lemniscate* is the locus of the intersection of a tangent to the equilateral hyperbola and a perpendicular to the tangent from the origin. (See fig. 13.)

17. Trace the curve $\rho = a \sin^3(\theta/3)$. (See fig. 14.)

18. Trace the curve $\rho = a \cos\theta \cos 2\theta$.

19. Trace the logarithmic spiral $\rho = e^{a\theta}$.

PART II. INTEGRAL CALCULUS.

CHAPTER I.

STANDARD FORMS. DIRECT INTEGRATION.

156. Integration. Having given the differential of a function, *integration* is the operation of finding the function. In other words, having given the ratio of the rate of a function to that of its variable, *integration* is the operation of finding the function.

A function is called an **integral** of its differential.

Thus, fx is an *integral* of $f'(x)\,dx$.

157. The **general** or **indefinite integral** of any differential $\phi(x)\,dx$ is the most general function whose differential is $\phi(x)\,dx$. The sign of *indefinite integration* is $\int$. Thus, $\int$ in the expression $\int\phi(x)\,dx$ denotes the *operation* of finding the indefinite integral of $\phi(x)\,dx$, while the whole expression denotes the indefinite integral itself.

For example, if C is a general constant, $x^3 + C$ is the most general function whose differential is $3\,x^2dx$; that is,

$$\int 3\,x^2dx \equiv x^3 + C.$$

x^3, $x^3 - 1$, $x^3 + 8$, $\cdots$ are *particular* integrals of $3\,x^2dx$.

The signs d and $\int$ indicate *inverse* operations, and, in general, neutralize each other.

Thus, $$d\int \phi(x)\,dx \equiv \phi(x)\,dx;$$

and $$\int d(fx + C) \equiv fx + C;$$

but $$\int d\,fx \equiv fx + C,$$

where C denotes a general constant called the *constant of integration.*

NOTATION. In the following pages we shall use $fx + C$ to denote the *general* or *indefinite* integral of the differential $\phi(x)\,dx$.

158. Elementary principles.

(i) $$\int \phi(u)\,du \equiv fu + C, \text{ if } d\,fu \equiv \phi(u)\,du.$$

This principle affords the simplest proof of formulas for indefinite integration.

Thus, $$\int u^n du \equiv \frac{u^{n+1}}{n+1} + C, \quad \text{since} \quad d\,\frac{u^{n+1}}{n+1} \equiv u^n du;$$

$$\int \frac{du}{u} \equiv \log u + C, \quad \text{since} \quad d(\log u) \equiv \frac{du}{u};$$

$$\int a^u du \equiv \frac{a^u}{\log a} + C, \quad \text{since} \quad d\,\frac{a^u}{\log a} \equiv a^u du.$$

In like manner all the formulas in § 159 can be proved.

(ii) *A constant factor can be transposed from one side of the sign of integration to the other without changing the value of the integral.*

For if a and c denote any constants, we have

$$a\,dy \equiv d(ay + ac);$$

$$\therefore \int a\,dy \equiv a(y + c) \equiv a\int d(y + c) \equiv a\int dy.$$

(iii) *The integral of a polynomial is equal to the sum of the integrals of its several terms.*

For
$$du + dy + dz \equiv d(u + y + z + C);$$
$$\therefore \int (du + dy + dz) \equiv u + y + z + C$$
$$\equiv \int du + \int dy + \int dz.$$

(iv) $\int 0 \equiv C$, since $dC \equiv 0$.

159. Standard forms and formulas. We give below a list of standard integrable forms. To integrate a differential directly, we reduce it to some one of these standard forms, and apply the formula. The most of these formulas are easily obtained by reversing the formulas for differentiation, and each is readily proved by (i) of § 158. The list will be gradually extended, and a supplementary list given later.

$$\int u^n du \equiv \frac{u^{n+1}}{n+1} + C, \text{ where } n \text{ is not } -1. \qquad [1]$$

$$\int \frac{du}{u} \equiv \log u + C. \qquad [2]$$

$$\int a^u du \equiv \frac{a^u}{\log a} + C. \qquad [3]$$

$$\int e^u du \equiv e^u + C. \qquad [4]$$

$$\int \sin u\, du \equiv - \cos u + C, \text{ or } \operatorname{vers} u + C'. \qquad [5]$$

$$\int \cos u\, du \equiv \sin u + C, \text{ or } - \operatorname{covers} u + C'. \qquad [6]$$

[illegible]

$$\int \sec^2 u\, du \equiv \tan u + C. \qquad [7]$$

$$\int \csc^2 u\, du \equiv - \cot u + C. \qquad [8]$$

$$\int \sec u \tan u\,du \equiv \sec u + C. \qquad [9]$$

$$\int \csc u \cot u\,du \equiv -\csc u + C. \qquad [10]$$

20.

$$\int \tan u\,du \equiv \log \sec u + C. \qquad [11]$$

$$\int \cot u\,du \equiv \log \sin u + C. \qquad [12]$$

$$\int \csc u\,du \equiv \log \tan \frac{u}{2} + C. \qquad [13]$$

$$\int \sec u\,du \equiv \log \tan \left(\frac{u}{2} + \frac{\pi}{4}\right) + C. \qquad [14]$$

$$\int \frac{du}{u^2 + a^2} \equiv \frac{1}{a} \tan^{-1} \frac{u}{a} + C, \text{ or } -\frac{1}{a} \cot^{-1} \frac{u}{a} + C'. \qquad [15]$$

$$\int \frac{du}{u^2 - a^2} \equiv \frac{1}{2a} \log \frac{u - a}{u + a} + C, \text{ or } \frac{1}{2a} \log \frac{a - u}{a + u} + C'. \qquad [16]$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} \equiv \sin^{-1} \frac{u}{a} + C, \text{ or } -\cos^{-1} \frac{u}{a} + C'. \qquad [17]$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} \equiv \log \left(u + \sqrt{u^2 \pm a^2}\right) + C. \qquad [18]$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} \equiv \frac{1}{a} \sec^{-1} \frac{u}{a} + C, \text{ or } -\frac{1}{a} \csc^{-1} \frac{u}{a} + C'. \qquad [19]$$

$$\int \frac{du}{\sqrt{2au - u^2}} \equiv \operatorname{vers}^{-1} \frac{u}{a} + C, \text{ or } -\operatorname{covers}^{-1} \frac{u}{a} + C'. \qquad [20]$$

Of these twenty forms only the first three are really *fundamental;* for by proper substitutions each of the others can be reduced to one of these three.

In the applications of these formulas we shall omit the *constant of integration* C, as it can readily be added when necessary.

160. *The variable parts of the indefinite integrals of the same or equal differentials are equal or differ by a constant.*

For if the differentials of two functions are equal, the rates of these functions will be equal; hence, the functions will be equal or differ by a constant.

For example, in formula [5] $\operatorname{vers} u$ and $-\cos u$ differ by 1; and in [17] $\sin^{-1}(u/a)$ and $-\cos^{-1}(u/a)$ differ by $\pi/2$.

161. Formulas [1] **and** [2] may be stated in words as follows:

The integral of a variable base with any constant exponent (except -1) into the differential of the base is the variable base with its exponent increased by 1, divided by the new exponent.

The integral of a fraction whose numerator is the differential of its denominator is the Naperian logarithm of the denominator.

Formula [1] fails to give a *finite* result when $n = -1$; but formula [2] provides for this case.

Formula [2] gives a *real* result only when u is positive.

EXAMPLES.

By § 158 and formula [1] or [2], find

1. $\int ax^6dx = a\int x^6dx = ax^7/7.$

The meaning of this result is that $ax^7/7 + C$ is the most general function which changes ax^6 times as fast as x, or that $y = ax^7/7 + C$ is the equation of the family of curves whose slope is ax^6.

2. $\int \frac{4\,dx}{x^5} = 4\int x^{-5}dx = -x^{-4}.$

Here $u = x$, $n = -5$, and $n + 1 = -4$.

3. $\int (x^2 - a^2)^{3/2}\,xdx = \frac{1}{2}\int (x^2 - a^2)^{3/2}\cdot 2\,xdx = (x^2 - a^2)^{5/2}/5.$

Here $n = 3/2$, $u = x^2 - a^2$; $\therefore du = 2\,xdx$.

Hence, we introduce the factor 2 before xdx and write its reciprocal $1/2$ on the left of the sign of integration.

4. $\int a(x+1)^2 dx = a(x+1)^3/3.$

5. $\int (\frac{7}{2}ax^{5/2} - \frac{5}{2}bx^{3/2})\,dx = \frac{7}{2}a\int x^{5/2}dx - \frac{5}{2}b\int x^{3/2}dx$
$= ax^{7/2} - bx^{5/2}.$

6. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}.$

7. $\int \left(bx^3 + \frac{1}{x^{3/2}}\right) dx = \frac{bx^4}{4} - \frac{2}{\sqrt{x}}.$

8. $\int \left(\frac{12}{x^3} - \frac{5}{x^4}\right) dx.$

9. $\int \frac{a\,dx}{x^n}.$

When the integral is not given, the result obtained by integration should be verified by principle (i) of § 158.

10. $\int b(6ax^2 + 8bx^3)^{5/3}(2ax + 4bx^2)\,dx$
$= \frac{b}{6}\int (6ax^2 + 8bx^3)^{5/3}(12ax + 24bx^2)\,dx$
$= b(6ax^2 + 8bx^3)^{8/3}/16.$

11. $\int [a(ax + bx^2)^{1/3}dx + 2b(ax + bx^2)^{1/3}x\,dx]$
$= \int (ax + bx^2)^{1/3}(a + 2bx)\,dx = \frac{3}{4}(ax + bx^2)^{4/3}.$

12. $\int (2a + 3bx)^3 dx.$ 13. $\int (1 + 9x/4)^{1/2} dx.$

14. $\int \frac{x^{n-1}dx}{(a + bx^n)^m} = \frac{1}{bn}\int (a + bx^n)^{-m}bnx^{n-1}dx = \frac{(a + bx^n)^{1-m}}{bn(1-m)}.$

15. $\int \frac{-(2ax - x^2)\,dx}{(3ax^2 - x^3)^{1/3}} = -(3ax^2 - x^3)^{2/3}/2.$

16. $\int \frac{x^2dx}{(a^2 + x^3)^{1/2}}.$

17. $\int \sqrt{2px}\,dx.$

18. $\int 2\pi y\left(\frac{y^2}{p^2} + 1\right)^{1/2} dy.$

19. $\int (2x^4 - 3x^2 + 1)^{1/2}(x^3 - 3x/4)\,dx.$

20. $\int \frac{dx}{x-a} = \int \frac{d(x-a)}{x-a} = \log(x-a).$

21. $\int \frac{x^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\int \frac{nbx^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\log(a + bx^n).$

Here $u = a + bx^n$; $\therefore du = nbx^{n-1}\,dx.$

22. $\int \frac{5\,x^2dx}{10\,x^3 + 15}.$

23. $\int \frac{x + 1}{x^2 + 2\,x}\,dx.$

24. $\int \frac{2\,bx^2dx}{ae + bx^3}.$

25. $\int \frac{5\,bxdx}{8\,a - 6\,bx^2}.$

26. $\int \frac{5\,(2\,a - x^2)^3\,dx}{x^5} = 5\left(-\frac{2\,a^3}{x^4} + \frac{6\,a^2}{x^2} + 6\,a\log x - \frac{x^2}{2}\right).$

Expand $(2\,a - x^2)^3$ and then integrate.

27. $\int (b - x^2)^3\,x^{1/2}\,dx = \frac{2}{3}\,b^3x^{3/2} - \frac{6}{7}\,b^2x^{7/2} + \frac{6}{11}\,bx^{11/2} - \frac{2}{15}\,x^{15/2}.$

28. $\int \frac{5\,x^3dx}{3\,x^4 + 7}.$

29. $\int (\log x)^3\,\frac{dx}{x}.$

30. $\int (\log x)^m\,\frac{dx}{x} = \frac{(\log x)^{m+1}}{m + 1}.$

31. $\int \frac{dx}{x\log x} = \log(\log x).$

32. $\int (a + b\log x)^{5/2}\,\frac{dx}{x} = \frac{2\,(a + b\log x)^{7/2}}{7\,b}.$

33. $\int \frac{dx}{a - x} = -\int \frac{d\,(a - x)}{a - x} = -\log(a - x).$

34. $\int (1 + x)\,(1 - x^2)\,xdx = x^2/2 - x^4/4 + x^3/3 - x^5/5.$

35. $\int \frac{x^2 + 1}{x - 1}\,dx = \int\left(x + 1 + \frac{2}{x - 1}\right)dx = \frac{1}{2}x^2 + x + 2\log(x - 1).$

36. $\int \frac{3\,x + 1}{3\,x - 1}\,dx = x + \log(3\,x - 1)^{2/3}.$

37. $\int \frac{x^3dx}{x + 1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(x + 1).$

38. $\int \frac{\log(2\,x + 3)}{2\,x + 3}\,dx = \frac{1}{4}[\log(2\,x + 3)]^2.$

39. $$\int \frac{(a^{2/3} - x^{2/3})^{3/4}}{x^{1/3}}\,dx = -\frac{3}{2}\int (a^{2/3} - x^{2/3})^{3/4}\left(-\frac{2}{3}x^{-1/3}\,dx\right)$$
$$= -\tfrac{6}{7}(a^{2/3} - x^{2/3})^{7/4}.$$

40. $$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \int (a^2x^{-2} - 1)^{-3/2}x^{-3}\,dx$$
$$= -\frac{1}{2\,a^2}\int (a^2x^{-2} - 1)^{-3/2}(-2\,a^2x^{-3}\,dx)$$
$$= \frac{(a^2x^{-2} - 1)^{-1/2}}{a^2} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

Sometimes, as above, we may transfer a variable factor from the base to the differential factor, or *vice versa*, and thus make the differential factor the differential of the base.

41. $$\int \frac{dx}{x^2\sqrt{x^2 + a^2}} = \int (1 + a^2x^{-2})^{-1/2}x^{-3}\,dx = -\frac{\sqrt{x^2 + a^2}}{a^2x}.$$

42. $$\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$$

43. $$\int \frac{dx}{(x^2 + a^2)^{3/2}}.$$

44. $$\int \frac{\sqrt{a^2 - x^2}}{x^4}\,dx = -\frac{(a^2 - x^2)^{3/2}}{3\,a^2x^3}.$$

45. $$\int \frac{\sqrt{x^2 - a^2}}{x^4}\,dx = \frac{(x^2 - a^2)^{3/2}}{3\,a^2x^3}.$$

46. $$\int \frac{x\,dx}{(2\,ax - x^2)^{3/2}} = \int (2\,ax^{-1} - 1)^{-3/2}x^{-2}\,dx = \frac{x}{a\sqrt{2\,ax - x^2}}.$$

47. $$\int \frac{\sqrt{2\,ax - x^2}}{x^3}\,dx.$$

48. $$\int \frac{dx}{x\sqrt{2\,ax - x^2}} = -\frac{\sqrt{2\,ax - x^2}}{ax}.$$

By § 158 and one or more of the formulas [1] to [4], find

49. $$\int a^{4x}\,dx = \frac{1}{4}\int a^{4x}\cdot 4\,dx = \frac{a^{4x}}{4\log a}.$$

Here u in [3] is $4\,x$; $\therefore du = 4\,dx$.

50. $$\int e^{x/n}\,dx = n\int e^{x/n}\frac{dx}{n} = ne^{x/n}.$$

51. $$\int ae^{bx}\,dx.$$

52. $$\int ca^{2x}\,dx.$$

53. $$\int \log c\cdot c^{x^4}x^3dx = \frac{\log c}{4}\int c^{x^4}\cdot 4\,x^3dx = c^{x^4}/4.$$

54. $\int (a^{nx} - b^{mx})\, dx = \dfrac{a^{nx}}{n \log a} - \dfrac{b^{mx}}{m \log b}.$

55. $\int (e^{u} - e^{-u})^2\, du = (e^{2u} - e^{-2u})/2 - 2u.$

56. $\int (2e^{3x} - a)^{3/2} e^{3x}\, dx = (2e^{3x} - a)^{5/2}/15.$

57. $\int \dfrac{e^x - 2}{e^x + 2}\, dx = \int \left(-dx + \dfrac{2e^x\, dx}{e^x + 2}\right) = 2 \log (e^x + 2) - x.$

58. $\int \dfrac{e^{2x}\, dx}{e^x + 1}.$

59. $\int b^x c^x dx = \dfrac{b^x c^x}{\log b + \log c}.$

162. *To obtain formulas* [11], [12], [13], *and* [14] *from* [2].

$$\int \tan u\, du \equiv \int \frac{\tan u \sec u\, du}{\sec u}$$

$$\equiv \log \sec u + C. \qquad [11]$$

$$\int \cot u\, du \equiv \int \frac{\cos u\, du}{\sin u}$$

$$\equiv \log \sin u + C. \qquad [12]$$

$$\int \csc u\, du \equiv \int \frac{du}{\sin u}$$

$$\equiv \int \frac{du}{2 \sin (u/2) \cos (u/2)}$$

$$\equiv \int \frac{\sec^2 (u/2)\, du/2}{\tan (u/2)}$$

$$\equiv \log \tan (u/2) + C. \qquad [13]$$

$$\int \sec u\, du \equiv \int \csc (u + \pi/2)\, du$$

$$\equiv \log \tan (u/2 + \pi/4) + C. \qquad [14]$$

EXAMPLES.

By § 158 and one or more of the formulas [1] to [14], find

1. $\int \cos mx dx = m^{-1} \int \cos mx \cdot m dx = m^{-1} \sin mx.$

Here u in [5] is mx; hence, $du = mdx$.

2. $\int \sec^2 mx dx = m^{-1} \tan mx.$

3. $\int \sin^4 x \cos x dx = \sin^5 x / 5.$

4. $\int \sec^2 x^3 \cdot x^2 dx = \tan x^3 / 3.$

5. $\int \cos x \sin x dx.$

6. $\int 7 \sec^2 x^2 \cdot x dx.$

7. $\int \cos (a + bx) dx.$

8. $\int \cos^4 3\theta \sin 3\theta d\theta = - \cos^5 3\theta / 15.$

9. $\int \sin^3 2x \cos 2x dx.$

10. $\int 5 \sec 3x \tan 3x dx.$

11. $\int 4 \csc ax \cot ax dx.$

12. $\int e^{\cos x} \sin x dx.$

13. $\int e^{2 \sin x} \cos x dx.$

14. $\int a^{\tan cx} \sec^2 cx dx.$

15. $\int \frac{\sin x dx}{a + b \cos x} = -\frac{1}{b} \int \frac{-b \sin x dx}{a + b \cos x} = -\frac{1}{b} \log (a + b \cos x).$

16. $\int \frac{(a + \cos \theta) d\theta}{a\theta + \sin \theta} = \log (a\theta + \sin \theta).$

17. $\int \sec u \csc u du = \int \frac{\sec^2 u du}{\tan u} = \log \tan u.$

18. $\int \sec^2 u \csc^2 u du = \int (\sec^2 u + \csc^2 u) du = \tan u - \cot u.$

19. $\int \sin^2 u du = \int (1/2 - \cos 2u/2) du = u/2 - \sin 2u/4.$

20. $\int \cos^2 u\,du = u/2 + \sin 2u/4.$

21. $\int \tan^2 u\,du = \tan u - u.$

22. $\int (\sec u - \tan u)^2\,du = 2(\tan u - \sec u) - u.$

23. $\int \frac{1-\sin u}{1+\sin u}\,du = \int \frac{(1-\sin u)^2}{\cos^2 u}\,du = 2(\tan u - \sec u) - u.$

24. $\int \frac{\cot u + \tan u}{\cot u - \tan u}\,du = \int \sec 2u\,du = \frac{1}{2}\log\tan\left(u + \frac{\pi}{4}\right).$

163. *To obtain formulus* [16] *and* [18] *from* [2].

$$\frac{1}{u^2-a^2} = \frac{1}{2a}\frac{1}{u-a} - \frac{1}{2a}\frac{1}{u+a}.$$

$$\therefore \int\frac{du}{u^2-a^2} = \frac{1}{2a}\int\frac{du}{u-a} - \frac{1}{2a}\int\frac{du}{u+a}$$

$$= \frac{1}{2a}\log\frac{u-a}{u+a} + C, \qquad (1)$$

or

$$\int\frac{du}{u^2-a^2} = \frac{1}{2a}\int\frac{-du}{a-u} - \frac{1}{2a}\int\frac{du}{a+u}$$

$$= \frac{1}{2a}\log\frac{a-u}{a+u} + C'. \qquad (2)$$

We use the form in (1) or (2) according as $u-a$ or $a-u$ is positive; that is, in each case we take the form which is real.

To obtain formula [18], let

$$\sqrt{u^2 \pm a^2} = z - u; \qquad (1)$$

then

$$\pm a^2 = z^2 - 2uz.$$

$$\therefore (z-u)\,dz = z\,du;$$

or

$$\frac{dz}{z} = \frac{du}{z-u} = \frac{du}{\sqrt{u^2\pm a^2}}. \qquad \text{by (1)}$$

$$\therefore \int\frac{du}{\sqrt{u^2\pm a^2}} = \int\frac{dz}{z} = \log z = \log(u+\sqrt{u^2\pm a^2}. \qquad \text{by (1)}$$

EXAMPLES.

1. $\int \frac{dx}{b^2x^2 + c^2} = \frac{1}{b}\int \frac{d(bx)}{(bx)^2 + c^2} = \frac{1}{bc}\tan^{-1}\frac{bx}{c}.$

2. $\int \frac{dx}{b^2x^2 - c^2} = \frac{1}{b}\int \frac{d(bx)}{(bx)^2 - c^2} = \frac{1}{2bc}\log\frac{bx - c}{bx + c}.$

3. $\int \frac{dx}{x^2 + 4x + 9} = \int \frac{dx}{(x+2)^2 + 5} = \frac{1}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}}.$

4. $\int \frac{dx}{x^2 + 4x + 1} = \int \frac{dx}{(x+2)^2 - 3} = \frac{1}{2\sqrt{3}}\log\frac{x + 2 - \sqrt{3}}{x + 2 + \sqrt{3}}.$

5. $\int \frac{xdx}{a^4 + x^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}.$

6. $\int \frac{x^2dx}{x^6 - 1} = \frac{1}{6}\log\frac{x^3 - 1}{x^3 + 1}.$

7. $\int \frac{dx}{16x^2 + 9}.$

8. $\int \frac{dx}{9x^2 - 4}.$

9. $\int \frac{dx}{c^2 - b^2x^2} = -\frac{1}{2bc}\log\frac{bx - c}{bx + c} = \frac{1}{2bc}\log\frac{bx + c}{bx - c}.$

10. $\int \frac{dx}{ax^2 + bx + c} = 2\int \frac{2adx}{(2ax + b)^2 + 4ac - b^2}$

$$= \frac{2}{\sqrt{4ac - b^2}}\tan^{-1}\frac{2ax + b}{\sqrt{4ac - b^2}}, \qquad (1)$$

$$\text{or } \frac{1}{\sqrt{b^2 - 4ac}}\log\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}. \qquad (2)$$

We use the form in (1) or (2) according as $4ac >$ or $< b^2$; that is, in any given case we take the form which is real.

11. $\int \frac{dx}{3x^2 + 4x + 7} = \int \frac{3dx}{(3x+2)^2 + 17} = \frac{1}{\sqrt{17}}\tan^{-1}\frac{3x + 2}{\sqrt{17}}.$

12. $\int \frac{2x + 7}{x^2 + 4x + 5}dx = \int \frac{(2x + 4)dx}{x^2 + 4x + 5} + 3\int \frac{dx}{(x+2)^2 + 1}$

$$= \log(x^2 + 4x + 5) + 3\tan^{-1}(x + 2).$$

13. $$\int\frac{(3x+2)\,dx}{2x^2+4x+5}=\frac{3}{4}\int\frac{(4x+4)\,dx}{2x^2+4x+5}-\int\frac{dx}{2x^2+4x+5} \qquad (1)$$
$$=\frac{3}{4}\log(2x^2+4x+5)-\frac{1}{\sqrt{6}}\tan^{-1}\frac{2x+2}{\sqrt{6}}.$$

In like manner any differential of the form $\frac{(px+q)\,dx}{ax^2+bx+c}$ can be integrated.

Note that the numerator of the first of the two fractions in the second member of (1) is the differential of its denominator.

14. $$\int\frac{x+1}{x^2+4x+5}\,dx=\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2).$$

15. $$\int\frac{2x-3}{5x^2+4}\,dx=\frac{1}{5}\log(5x^2+4)-\frac{3}{2\sqrt{5}}\tan^{-1}\frac{x\sqrt{5}}{2}.$$

16. $$\int\frac{x\,dx}{x^2+2x+1}=\log(x+1)+\frac{1}{x+1}.$$

17. $$\int\frac{dx}{\sqrt{a^2c^2-b^2x^2}}=\frac{1}{b}\int\frac{b\,dx}{\sqrt{(ac)^2-(bx)^2}}=\frac{1}{b}\sin^{-1}\frac{bx}{ac}.$$

18. $$\int\frac{dx}{\sqrt{b^2x^2\pm a^2c^2}}=\frac{1}{b}\log\left(bx+\sqrt{b^2x^2\pm a^2c^2}\right).$$

19. $$\int\frac{dx}{\sqrt{1-x-x^2}}=\int\frac{dx}{\sqrt{5/4-(x+1/2)^2}}=\sin^{-1}\frac{2x+1}{\sqrt{5}}.$$

20. $$\int\frac{dx}{\sqrt{ax^2-b}}=\frac{1}{\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2-b}\right).$$

21. $$\int\frac{dx}{\sqrt{x^2+2ax}}=\log\left(x+a+\sqrt{x^2+2ax}\right).$$

22. $$\int\frac{x^{1/2}\,dx}{\sqrt{8-4x^3}}=\frac{1}{3}\int\frac{3x^{1/2}\,dx}{\sqrt{(2\sqrt{2})^2-(2x^{3/2})^2}}=\frac{1}{3}\sin^{-1}\sqrt{\frac{x^3}{2}}.$$

23. $$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{2a\,dx}{\sqrt{(2ax+b)^2+4ac-b^2}}$$
$$=\frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right).$$

24. $$\int\frac{dx}{\sqrt{-ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{2\,adx}{\sqrt{4\,ac+b^2-(2\,ax-b)^2}}$$
$$=\frac{1}{\sqrt{a}}\sin^{-1}\frac{2\,ax-b}{\sqrt{4\,ac+b^2}}.$$

25. $$\int\frac{dx}{\sqrt{4+3\,x-2\,x^2}}=\frac{1}{\sqrt{2}}\int\frac{4\,dx}{\sqrt{41-(4\,x-3)^2}}=\frac{1}{\sqrt{2}}\sin^{-1}\frac{4\,x-3}{\sqrt{41}}.$$

26. $$\int\frac{dx}{\sqrt{2\,x^2+3\,x+4}}=\frac{1}{\sqrt{2}}\log\left(4\,x+3+2\sqrt{2}\sqrt{2\,x^2+3\,x+4}\right).$$

27. $$\int\frac{dx}{\sqrt{3\,x-x^2-2}}.$$

28. $$\int\frac{dx}{\sqrt{1+x^2+x}}.$$

29. $$\int\frac{dx}{\sqrt{2\,x^2+2\,x+3}}.$$

30. $$\int\frac{x\,dx}{\sqrt{a^4-x^4}}.$$

31. $$\int\frac{dx}{x\sqrt{c^2x^2-a^2b^2}}=\int\frac{cdx}{cx\sqrt{(cx)^2-(ab)^2}}=\frac{1}{ab}\sec^{-1}\frac{cx}{ab}.$$

32. $$\int\frac{dx}{x\sqrt{b^2x^2-a^2}}.$$

33. $$\int\frac{5\,dx}{x\sqrt{3\,x^2-5}}.$$

34. $$\int\frac{dx}{\sqrt{7\,x^4-5\,x^2}}=\int\frac{\sqrt{7}\,dx}{x\sqrt{7}\cdot\sqrt{7\,x^2-5}}=\frac{1}{\sqrt{5}}\sec^{-1}\frac{x\sqrt{7}}{\sqrt{5}}.$$

35. $$\int\frac{-\,dx}{\sqrt{8\,cx-c^2x^2}}=\frac{1}{c}\int\frac{-\,cdx}{\sqrt{8\cdot cx-(cx)^2}}=\frac{1}{c}\,\text{covers}^{-1}\frac{cx}{4}.$$

36. $$\int\frac{dx}{\sqrt{2\,abx-b^2x^2}}.$$

37. $$\int\frac{dx}{\sqrt{ax-x^2}}.$$

38. $$\int\frac{(2\,x+3)\,dx}{\sqrt{x^2+x+1}}=\int\frac{(2\,x+1)\,dx}{\sqrt{x^2+x+1}}+2\int\frac{dx}{\sqrt{x^2+x+1}}\qquad(1)$$
$$=2\sqrt{x^2+x+1}+2\log\left(2\,x+1+2\sqrt{x^2+x+1}\right).$$

In like manner any differential of the form $\dfrac{(px+q)\,dx}{\sqrt{ax^2+2\,bx+c}}$ can be integrated.

Note that the numerator of the first of the two fractions in the second member of (1) is the differential of the *base* in the denominator.

39. $$\int \frac{-x\,dx}{\sqrt{cx-x^2}} = \int \frac{c-2x-c}{2\sqrt{cx-x^2}}dx$$
$$= \sqrt{cx-x^2} - \frac{c}{2}\operatorname{vers}^{-1}\frac{2x}{c}.$$

40. $$\int \frac{(x^2-a^2)^{1/2}\,dx}{x} = \int \frac{(x^2-a^2)\,dx}{x\sqrt{x^2-a^2}}$$
$$= \int \frac{x\,dx}{\sqrt{x^2-a^2}} - \int \frac{a^2dx}{x\sqrt{x^2-a^2}}$$
$$= \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}.$$

41. $$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}dx = \sin^{-1}x - \sqrt{1-x^2}.$$

42. $$\int \frac{\sqrt{x+a}}{x\sqrt{x-a}}dx = \sec^{-1}\frac{x}{a} + \log(x+\sqrt{x^2-a^2}).$$

CHAPTER II.

DEFINITE INTEGRALS. APPLICATIONS.

164. Definite and corrected integrals. Let $b > a$; then the increment produced in the indefinite integral $fx + C$ by the increase of x from a to b is

$$fb + C - (fa + C) \equiv fb - fa.$$

This increment of the indefinite integral of $\phi(x)\,dx$ is called "the **definite** integral of $\phi(x)\,dx$ between the limits a and b," and is denoted by $\int_a^b \phi(x)\,dx$.

Hence, $$\int_a^b \phi(x)\,dx \equiv fb - fa. \qquad (1)$$

The symbol $\int_a^b$ in the expression $\int_a^b \phi(x)\,dx$ denotes the operation of finding the increment of the indefinite integral of $\phi(x)\,dx$ from $x = a$ to $x = b$. b is called the 'upper' or 'superior' limit, and a the 'lower' or 'inferior' limit.

The limits a and b must be so chosen that ϕx will be finite, continuous, and of the same quality, from $x = a$ to $x = b$.

The expression $fb - fa$ is often denoted by $fx\Big]_a^b$.

If, in (1), we make the upper limit variable and put x in the place of b, we obtain

$$\int_a^x \phi(x)\,dx = fx - fa, \text{ or } fx\Big]_a^x, \qquad (2)$$

which is called "the **corrected** integral of $\phi(x)\,dx$."

By $\int_a^\infty \phi(x)\,dx$ is meant the limit of $\int_a^x \phi(x)\,dx$ when $x = \infty$.

EXAMPLES.

1. $\displaystyle\int_1^x x^3 dx = \frac{x^4}{4}\Big]_1^x = \frac{x^4}{4} - \frac{1}{4}.$

2. $\displaystyle\int_a^b nx\,dx = \frac{n}{2}x^2\Big]_a^b = \frac{n}{2}(b^2 - a^2).$

3. $\displaystyle\int_1^x \left(\frac{dx}{x} - \frac{dx}{2-x}\right) = \log(2x - x^2)\Big]_1^x = \log(2x - x^2).$

4. $\displaystyle\int_0^\infty \frac{x\,dx}{1 + x^4} = \frac{1}{2}\tan^{-1}x^2\Big]_0^\infty = \frac{\pi}{4}.$

5. $\displaystyle\int_0^2 6x^3 dx = 24.$

6. $\displaystyle\int_0^a (ax^2 - x^3)\,dx = \frac{a^4}{12}.$

7. $\displaystyle\int_0^a \frac{dx}{a^2 + x^2} = \frac{\pi}{4a}.$

8. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}.$

9. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta\,d\theta}{\cos^2\theta} = \sqrt{2} - 1.$

10. $\displaystyle\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}.$

11. $\displaystyle\int_0^{2r} \frac{2\sqrt{2r}\,dy}{\sqrt{2r - y}} = 8r.$

12. $\displaystyle\int_{-b}^b \frac{\pi}{a^4}(y^2 - b^2)^4\,dy = \frac{256\,\pi b^9}{315\,a^4}.$

13. $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}.$

14. $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^3 x\,dx = \frac{1}{12}.$

15. $\displaystyle\int_0^x \frac{dx}{\sqrt{5 - 2x - x^2}} = \sin^{-1}\frac{x+1}{\sqrt{6}}\Big]_0^x = \sin^{-1}\frac{x+1}{\sqrt{6}} - \sin^{-1}\frac{1}{\sqrt{6}}.$

16. $\displaystyle\int_1^x \frac{dx}{3 + 2x + x^2} = \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{x+1}{\sqrt{2}} - \tan^{-1}\sqrt{2}\right).$

17. $\displaystyle\int_0^x \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\,dx = \frac{1}{e}\log(x^e + e^x).$

165. *Corresponding definite or corrected integrals of equal differentials are equal.*

For the corresponding increments of variables which change at equal rates are equal.

Ex. $\int_4^6 \frac{3x-1}{(x-3)^2}dx = 3\log 3 + \frac{16}{3}$.

Let $z = x - 3$; then $x = z + 3$, $dx = dz$, and

$$\frac{3x-1}{(x-3)^2}dx = \frac{3z+8}{z^2}dz.$$

When $x = 4$, $z = 1$, and when $x = 6$, $z = 3$;

$$\therefore \int_4^6 \frac{3x-1}{(x-3)^2}dx = \int_1^3 \frac{3z+8}{z^2}dz \qquad \S\ 165$$

$$= \Big[3\log z - 8/z\Big]_1^3$$

$$= 3\log 3 + 16/3.$$

This example and those which follow illustrate also how the introduction of a new variable often simplifies a given differential and renders it directly integrable.

EXAMPLES.

1. $\int_4^6 \frac{x^2-2x}{(x-3)^3}dx = \int_1^3 \frac{z^2+4z+3}{z^3}dz = \log 3 + 4$, where $z = x - 3$.

2. $\int_0^2 \frac{x^3dx}{(x^2+1)^{2/3}} = \int_1^5 \frac{(z-1)\,dz}{2z^{2/3}} = \frac{3}{8}(\sqrt[3]{5}+3)$, where $z = x^2+1$.

3. $\int_1^a \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\int_a^1 \frac{dz}{\sqrt{z^2-1}}$

$= \log(a + \sqrt{a^2-1})/a$, where $z = a/x$.

4. $\int_0^1 \frac{e^{2x}dx}{(e^x+1)^{1/4}} = \int_2^{e+1} \frac{(z-1)\,dz}{z^{1/4}}$

$= \frac{4}{21}[(3e-4)(e+1)^{3/4} + \sqrt[4]{8}]$, where $z = e^x + 1$.

5. $\int_1^a \frac{dx}{x\sqrt{a^2+x^2}} = \frac{1}{a}\log\frac{a+\sqrt{a^2+1}}{1+\sqrt{2}}$.

In example 5 put $x = a/z$, in example 6 put $\theta + a = z$.

6. $\int_{-2a}^{b-a} \frac{\cos\theta\, d\theta}{\cos(\theta+a)} = (b+a)\cos a - \sin a \log(\cos b/\cos a)$.

166. Geometric meaning of $\int \phi(x)dx$, $\int_a^b \phi(x)dx$, $\int_a^x \phi(x)dx$.

Let SPQ be the locus of $y = \phi x$.

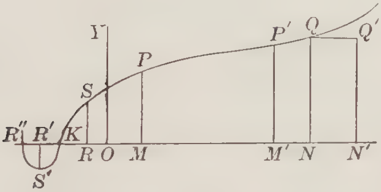
Conceive the variable ordinate y or NQ to trace the area between the x-axis and $y = \phi x$ as the point (x, y) moves along the curve to the right.

Let A denote the area bounded by the x-axis, $y = \phi x$, some undetermined fixed ordinate, as RS or $R'S'$, and the moving ordinate NQ.

Let $dx = NN'$; then $dA = NQQ'N'$;

$$\therefore dA = y dx = \phi(x)\, dx. \tag{1}$$

$$\therefore \int y dx = \int \phi(x)\, dx = A. \tag{2}$$



Let $OM = a$, and $OM' = b$; then the increment produced in A by the increase of x from a to b is $MPP'M'$;

$$\therefore \int_a^b y dx = \int_a^b \phi(x)\, dx = \left\{ \begin{array}{l} \text{area bounded by the } x\text{-axis,} \\ y = \phi x,\ x = a, \text{ and } x = b. \end{array} \right\} \tag{3}$$

Similarly, if $OM = a$ and $ON = x$, we have

$$\int_a^x y dx = \int_a^x \phi(x)\, dx = \left\{ \begin{array}{l} \text{area bounded by the } x\text{-axis,} \\ y = \phi x,\ x = a, \text{ and the ordinate } y. \end{array} \right\} \tag{4}$$

In (2), A is indeterminate so long as the fixed ordinate RS, or $R'S'$, is indeterminate.

From (1), by Cor. 1 of § 12, we know that the areas in (3) and (4) will be positive or negative according as ϕx is positive or negative from $x = a$ to $x = b$.

Hence, if a curve crosses the x-axis the area above, and the area below, this axis must be obtained separately.

For example, to find the area bounded by $y = \phi x$, the x-axis, and $M'P'$, we find the areas $R''S'K$ and $KPP'M'$ separately and take their arithmetical sum.

From the geometrical meaning of an integral it follows that $\phi(x)\,dx$ has an integral whenever ϕx is continuous.

Ex. Give the geometric meaning of the definite integral in each of the examples on page 159.

EXAMPLES.

1. Find the area bounded by the x-axis and $y = x^3 + ax^2$.

The curve cuts the x-axis at $(-a, 0)$ and $(0, 0)$; hence, the limits are $-a$ and 0.

Here $dA = y\,dx = (x^3 + ax^2)\,dx$;

$$\therefore \text{area} = \int_{-a}^{0} (x^3 + ax^2)\,dx = \Big[x^4/4 + ax^3/3\Big]_{-a}^{0} \qquad \text{by (3)}$$
$$= a^4/12.$$

In each problem the reader should first gain a clear idea of the boundary of the figure whose area is required.

2. Find the area bounded by the x-axis, the parabola $x^2 + 4y = 0$, and the line $x = 4$. *Ans.* $16/3$.

Since the area lies below the x-axis, the formula gives a negative expression for it.

3. Find the area bounded by the x-axis, the curve $y = x^3$, and the lines $x = -2$ and $x = 2$. *Ans.* 8.

Here we find the area below, and the area above, the x-axis separately, and take their arithmetical sum.

4. Find the area bounded by the curve $y = e^x$, the x-axis, the y-axis, and the line $x = h$. *Ans.* $e^h - 1$.

5. Show that the area bounded by $y = x - x^3$ and $y = 0$ is $1/2$.

6. Find the area bounded by the parabola $y^2 = 4px$ and any chord perpendicular to its axis. *Ans.* $4xy/3$.

Here the area required is twice the area bounded by the x-axis, $y = 2p^{1/2}x^{1/2}$, and the ordinate y.

$$\therefore \text{area} = 2\int_0^x 2p^{1/2}x^{1/2}dx = (2/3)\,2xy; \qquad \text{by (4)}$$

that is, the area is $2/3$ the circumscribed rectangle.

7. Find the area bounded by the witch $y(x^2 + 4a^2) = 8a^3$ and its asymptote $y = 0$.

$$\text{Area} = 2\int_0^\infty \frac{8a^3dx}{x^2 + 4a^2} = 8a^2 \tan^{-1}\frac{x}{2a}\Big]_0^\infty = 4\pi a^2.$$

Here the area between the curve and its asymptote is finite.

8. Find the area bounded by the hyperbola $xy = 1$, its asymptote $y = 0$, and the lines $x = 1$ and $x = a$. *Ans.* $\log a$.

When $a = \infty$, $\log a = \infty$; hence, the area between the hyperbola and an asymptote is infinite.

9. Find the area bounded by the curve $y = \cos x$, the x-axis, the y-axis, and $x = \pi$. *Ans.* 2.

10. Show that the area bounded by the curve $y = \tan x$, the x-axis, and the asymptote $x = \pi/2$ is infinite.

167. Formulas for accelerated motion. Let t denote a portion of time, s the distance traversed by a moving body, v the velocity, and a the acceleration; then we have the following formulas:

(i) $v = ds/dt; \quad \therefore s = \int vdt, \; t = \int ds/v.$

(ii) $a = dv/dt; \quad \therefore v = \int adt, \; t = \int dv/a.$

EXAMPLES.

1. To find the fundamental formulas for uniformly accelerated motion.

Let v_0 and s_0 denote, respectively, the *initial* velocity and distance; that is, the values of v and s when $t = 0$; and let α' denote the constant acceleration. We then have

$$v = v_0 + \int_0^t \alpha' dt = \alpha' t + v_0, \tag{1}$$

$$s = s_0 + \int_0^t v dt = \alpha' t^2/2 + v_0 t + s_0. \tag{2}$$

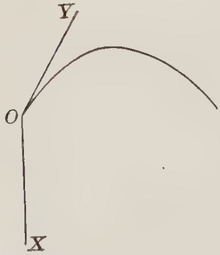
If $v = 0$ and $s = 0$ when $t = 0$, (1) and (2) become

$$v = \alpha' t, \qquad s = \alpha' t^2/2; \tag{3}$$

$$\therefore t = \sqrt{2s/\alpha'}, \quad v = \sqrt{2\alpha' s}. \tag{4}$$

The acceleration produced by gravity at the earth's surface is about 32.17 ft. a second, and is usually represented by g. Substituting g for α' in equalities (3) and (4), we obtain the four formulas for the *free fall of bodies in a vacuum* near the earth's surface.

2. A rifle ball is projected from O in the direction OY with a velocity of c feet a second. Find its path, knowing that its velocity acquired in t seconds along the action line of gravity OX is gt feet a second.



Let OX and OY be the co-ordinate axes; then $dy/dt = c, \qquad dx/dt = gt;$

$$\therefore y = \int_0^t c dt = ct, \qquad x = gt^2/2. \tag{1}$$

Eliminating t between equations (1), we have

$$y^2 = 2c^2 x/g.$$

Hence, the path of the ball is an arc of a parabola.

3. A body starts from O, and in t seconds its velocity in the direction of OX is $2act$, and in the direction of OY it is $a^2t^2 - c^2$; find its velocity along its path Onm, the distances in the direction of each axis and along the line of its path, and the equation of its path, the axes being rectangular.

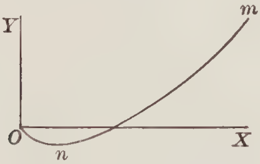
Let OX and OY be the axes, and let s denote the length of the path Onm; then

$$\frac{dx}{dt} = 2\,act; \qquad \therefore x = \int_0^t 2\,act\,dt = act^2; \tag{1}$$

$$\frac{dy}{dt} = a^2t^2 - c^2; \quad \therefore y = \int_0^t (a^2t^2 - c^2)\,dt = a^2t^3/3 - c^2t; \tag{2}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a^2t^2 + c^2; \tag{3}$$

$$\therefore s = \int_0^t (a^2t^2 + c^2)\,dt = a^2t^3/3 + c^2t. \tag{4}$$



Eliminating t between (1) and (2), we obtain

$$y = \left(\frac{ax}{3c} - c^2\right)\sqrt{\frac{x}{ac}}, \tag{5}$$

which is the equation of the path Onm.

4. Given $v = ft$, to represent the time, the velocity, the distance, and the acceleration geometrically.

Construct the locus of $v = ft$, t being represented by abscissas and v by ordinates; then the distance will be represented by the area between $v = ft$ and the x-axis, and the acceleration by the line representing dv when dt is represented by a unit length.

5. A body is projected upwards with a velocity of 80 feet per second; find in what time it will return to the place of starting.

Ans. 5 seconds, nearly.

6. From a balloon which is ascending at a uniform velocity of 20 feet per second two balls are dropped, one of them 3 seconds before the other; find how far apart they will be 5 seconds after the first one was dropped.

Ans. 398 feet, nearly.

168. Change of limits. The following formulas are readily proved from the definition of a definite integral:

$$\int_a^b \phi(x)\,dx \equiv -\int_b^a \phi(x)\,dx; \tag{i}$$

for each member $\equiv fb - fa$, if $dfx = \phi(x)\,dx$.

$$\int_a^b \phi(x)\,dx \equiv \int_a^c \phi(x)dx + \int_c^b \phi(x)\,dx; \tag{ii}$$

for the second member $\equiv fc - fa + fb - fc \equiv fb - fa$.

$$\int_0^a \phi(x)dx \equiv \int_0^a \phi(a-x)\,dx; \tag{iii}$$

for the second member $\equiv -\int_0^a \phi(a-x)\,d(a-x)$

$$\equiv -f(a-x)\Big]_0^a \equiv fa - f0.$$

NOTE. The Integral Calculus was invented to obtain the areas of curvilinear figures, or for quadrature, as it is often called.

CHAPTER III.

INTEGRATION OF RATIONAL FRACTIONS.

169. Decomposition of fractions. When the numerator of a rational fraction is not of a lower degree than its denominator, the fraction, said to be *improper*, should be reduced to a mixed expression before integration. For example,

$$\frac{x^4}{x^3 + 2x^2 - x - 2}\,dx \equiv (x-2)\,dx + \frac{5x^2 - 4}{x^3 + 2x^2 - x - 2}\,dx.$$

The numerator of this new rational fraction is of a lower degree than its denominator. Such a fraction, called a *proper* fraction, if not directly integrable, can be *decomposed into partial real fractions* which are integrable. These partial fractions will differ in form, according as the *simplest real* factors of the denominator of the given fraction are :

I. Linear and unequal.
II. Linear and equal.
III. Quadratic and unequal.
IV. Quadratic and equal.

To present the subject in the simplest manner we solve below particular examples in each of the four cases.

170. Case I. To each of the unequal linear factors of the denominator as $x - a$, there will correspond a partial fraction of the form $A/(x-a)$.

Ex. $\displaystyle\int \frac{2x+3}{x^3 + x^2 - 2x}\,dx = \log\frac{(x-1)^{5/3}}{x^{3/2}(x+2)^{1/6}}.$

The denominator $x^3 + x^2 - 2x \equiv x(x-1)(x+2)$.

Hence, by addition of fractions we know that the given fraction is the sum of three fractions whose denominators are x, $x-1$, and $x+2$, respectively, and whose numerators do not involve x.

We therefore assume

$$\frac{2x+3}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}, \tag{1}$$

in which A, B, and C are unknown constants.

Clearing (1) of fractions, we obtain

$$2x+3 = A(x-1)(x+2) + B(x+2)x + C(x-1)x \tag{2}$$

$$= (A+B+C)x^2 + (A+2B-C)x - 2A. \tag{3}$$

Equating the coefficients of like powers of x in (3), we have

$$A+B+C=0,\ A+2B-C=2,\ -2A=3. \tag{4}$$

Solving system (4), we obtain

$$A=-3/2,\ B=5/3,\ C=-1/6.$$

Substituting these values in (1), we obtain the identity,

$$\frac{2x+3}{x^3+x^2-2x} = -\frac{3}{2x} + \frac{5}{3(x-1)} - \frac{1}{6(x+2)};$$

$$\therefore \int \frac{(2x+3)\,dx}{x^3+x^2-2x} = -\frac{3}{2}\int\frac{dx}{x} + \frac{5}{3}\int\frac{dx}{x-1} - \frac{1}{6}\int\frac{dx}{x+2}$$

$$= -\tfrac{3}{2}\log x + \tfrac{5}{3}\log(x-1) - \tfrac{1}{6}\log(x+2).$$

The values of A, B, and C may be obtained directly from (2) as follows:

Making $x=0$, (2) becomes $3=-2A$; $\therefore A=-3/2$.
Making $x=1$, (2) becomes $5=3B$; $\therefore B=5/3$.
Making $x=-2$, (2) becomes $-1=6C$; $\therefore C=-1/6$.

EXAMPLES.

1. $\displaystyle\int\frac{x^2-1}{x^2-4}\,dx = x + \frac{3}{4}\log\frac{x-2}{x+2}.$

2. $\displaystyle\int\frac{x^2+x-1}{x^3+x^2-6x}\,dx = \log\left[x^{1/6}(x-2)^{1/2}(x+3)^{1/3}\right].$

3. $\displaystyle\int\frac{x-1}{x^2+6x+8}\,dx.$

4. $\displaystyle\int\frac{5x+1}{x^2+x-2}\,dx.$

5. $\displaystyle\int\frac{(x^2+1)\,dx}{(2x+1)(x^2-1)} = \frac{1}{6}\log\frac{(x+1)^6(x-1)^2}{(2x+1)^5}.$

4 6 7 1 9 8 7 9 1 8 4 3 2 7 1 6 5 0

1
3
1
1
2

6. $\int \frac{x^5+x^4-8}{x^3-4x}\,dx = \frac{x^3}{3}+\frac{x^2}{2}+4x+\log\frac{x^2(x-2)^5}{(x+2)^3}.$

7. $$\int \frac{x\,dx}{x^2-6x+7} = \frac{3+\sqrt{2}}{2\sqrt{2}}\log(x-3-\sqrt{2})-\frac{3-\sqrt{2}}{2\sqrt{2}}\log(x-3+\sqrt{2})$$
$$=\frac{1}{2}\log(x^2-6x+7)+\frac{3}{2\sqrt{2}}\log\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}.$$

8. $\int \frac{x\,dx}{(x-a)(x-b)}.$

9. $\int \frac{x^2\,dx}{(x+1)(x^2+x-6)}.$

171. Case II. To r equal linear factors of the denominator as $(x-b)^r$, there will correspond a series of r partial fractions of the form $\frac{A}{(x-b)^r}+\frac{B}{(x-b)^{r-1}}+\cdots+\frac{L}{x-b}.$

Ex. 1. $\int \frac{3x^2-7x+6}{(x-1)^3}\,dx = 3\log(x-1)+\frac{x-2}{(x-1)^2}.$

Expressing the numerator in powers of $(x-1)$, we obtain

$$3x^2-7x+6 \equiv 3(x-1)^2-(x-1)+2; \tag{1}$$

for $3(x-1)^2 \equiv \underline{3x^2-6x+3}$

and $\qquad\qquad\quad -\ \ x+3 \equiv -(x-1)+2.$

Dividing both members of (1) by $(x-1)^3$, we obtain

$$\frac{3x^2-7x+6}{(x-1)^3} \equiv \frac{3}{x-1}-\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}. \tag{2}$$

$$\therefore \int \frac{3x^2-7x+6}{(x-1)^3}\,dx = 3\int\frac{dx}{x-1}-\int\frac{dx}{(x-1)^2}+2\int\frac{dx}{(x-1)^3}$$
$$=3\log(x-1)+(x-1)^{-1}-(x-1)^{-2}.$$

From (2) we see that to resolve the given fraction into partial fractions by undetermined coefficients, we should assume

$$\frac{3x^2-7x+6}{(x-1)^3} = \frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1},$$

and proceed as in § 170.

Ex. 2. $$\int \frac{dx}{(x-1)^2(x+1)} = \log\left(\frac{x+1}{x-1}\right)^{1/4} - \frac{1}{2(x-1)}. \tag{1}$$

Here both Case I and Case II are involved; hence, we assume

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}. \tag{2}$$

Clearing (2) of fractions, equating the coefficients of the like powers of x, and solving the resulting system, we obtain $A = 1/2$, $B = -1/4$, $C = 1/4$. Substituting these values in (2) and integrating, we obtain (1).

EXAMPLES.

1. $$\int \frac{3x-4}{(x+2)^2}\,dx = 3\log(x+2) + \frac{10}{x+2}.$$

2. $$\int \frac{2x^2-3x+4}{(x-3)^3}\,dx = 2\log(x-3) - \frac{18x-41}{2(x-3)^2}.$$

3. $$\int \frac{4x^2-6x+7}{(2x+3)^3}\,dx = \frac{1}{2}\log(2x+3) + \frac{9}{2(2x+3)} - \frac{25}{4(2x+3)^2}.$$

4. $$\int_1^x \frac{(2x-5)\,dx}{(x+3)(x+1)^2} = \frac{7}{2(x+1)} + \frac{11}{4}\log\frac{2(x+1)}{x+3} - \frac{7}{4}.$$

5. $$\int_0^x \frac{x^2\,dx}{x^3+5x^2+8x+4} = \frac{4}{x+2} + \log(x+1) - 2.$$

6. $$\int \frac{(2x^3+7x^2+6x+2)\,dx}{x^4+3x^3+2x^2} = \log\left[x(x+1)\left(\frac{x}{x+2}\right)^{1/2}\right] - \frac{1}{x}.$$

7. $$\int \frac{x^2\,dx}{(x-1)^3(x+1)}.$$

8. $$\int \frac{(x+1)\,dx}{(x-1)^2(x+2)^2}.$$

172. Case III. To each of the unequal quadratic factors of the denominator as x^2+px+q, there will correspond a partial fraction of the form $\dfrac{Ax+B}{x^2+px+q}$.

Ex. 1. $$\int \frac{(x^2+1)\,dx}{(x^2+2)(2x^2+1)} = \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{3\sqrt{2}}\tan^{-1}x\sqrt{2}. \tag{1}$$

By addition of fractions we know that the given fraction is the sum of two real fractions whose denominators are x^2+2 and $2x^2+1$ respectively, and whose numerators cannot be above the first degree in x.

Hence, we assume

$$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{2x^2+1}. \qquad (2)$$

Clearing (2) of fractions, we obtain

$$x^2+1 = (2A+C)x^3 + (2B+D)x^2 + (A+2C)x + B + 2D.$$

$$\therefore 2A+C=0,\ 2B+D=1,\ A+2C=0,\ B+2D=1.$$

$$\therefore A=C=0,\ B=D=1/3.$$

Substituting these values in (2) and integrating, we obtain (1).

Ex. 2. $$\int \frac{x\,dx}{(x+1)(x^2+1)} = \frac{1}{2}\log\frac{(x^2+1)^{1/2}}{x+1} + \frac{1}{2}\tan^{-1}x. \qquad (1)$$

Here both Case I and Case III are involved; hence, we assume

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}. \qquad (2)$$

Proceeding as above, we find $A=-1/2$, $B=C=1/2$. Substituting in (2) and integrating, we obtain (1).

EXAMPLES.

1. $$\int \frac{x^2\,dx}{x^4+x^2-2} = \frac{1}{6}\log\frac{x-1}{x+1} + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$$

2. $$\int \frac{x^3\,dx}{x^4+3x^2+2}.$$

3. $$\int \frac{x^2\,dx}{1-x^4}.$$

4. $$\int \frac{x^2\,dx}{(x-1)^2(x^2+1)} = -\frac{1}{2(x-1)} + \frac{1}{4}\log\frac{x^2-2x+1}{x^2+1}.$$

5. $$\int \frac{dx}{x^3+1} = \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{(x-2)\,dx}{x^2-x+1};$$

$$\int \frac{(x-2)\,dx}{x^2-x+1} = \frac{1}{2}\int\frac{(2x-1)\,dx}{x^2-x+1} - \frac{1}{2}\int\frac{3\,dx}{x^2-x+1};$$

$$\therefore \int \frac{dx}{x^3+1} = \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

6. $$\int \frac{dx}{1-x^3} = \frac{1}{6}\log\frac{x^2+x+1}{(x-1)^2} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

7. $\displaystyle\int\frac{dx}{x^4+8x^2-9}\cdot$ 8. $\displaystyle\int\frac{x^2+1}{x^4-x^2+1}\,dx.$

9. $$\int\frac{(5x^2-1)\,dx}{(x^2+3)(x^2-2x+5)}$$
$$=\log\frac{x^2-2x+5}{x^2+3}+\frac{5}{2}\tan^{-1}\frac{x-1}{2}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\cdot$$

10. $$\int\frac{dx}{(1+x)^2(1+2x+4x^2)}$$
$$=\frac{1}{3}\log\frac{(1+x)^2}{1+2x+4x^2}-\frac{1}{3(1+x)}+\frac{2}{3\sqrt{3}}\tan^{-1}\frac{4x+1}{\sqrt{3}}\cdot$$

11. $$\int\frac{x^2+3x+1}{x^4+x^2+1}\,dx=\frac{4}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\cdot$$

12. $$\int\frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{a(b^2-a^2)}\tan^{-1}\frac{x}{a}+\frac{1}{b(a^2-b^2)}\tan^{-1}\frac{x}{b}\cdot$$

173. Case IV. To r equal quadratic factors of the denominator as $(x^2+px+q)^r$, there will correspond r partial fractions of the form

$$\frac{Ax+B}{(x^2+px+q)^r}+\frac{Cx+D}{(x^2+px+q)^{r-1}}+\cdots+\frac{Lx+M}{x^2+px+q}\cdot$$

Ex. $$\int\frac{2x^3+x^2+3x+2}{(x^2+1)^2}\,dx$$
$$=\log(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{x-1}{2(x^2+1)}\cdot\qquad(1)$$

Expressing the numerator in powers of (x^2+1), we obtain

$$2x^3+x^2+3x+2\equiv(x^2+1)(2x+1)+(x+1).$$
$$\frac{2x^3+x^2+2x+1}{x+1}$$

$$\therefore\int\frac{2x^3+x^2+3x+2}{(x^2+1)^2}\,dx=\int\frac{2x+1}{x^2+1}\,dx+\int\frac{x+1}{(x^2+1)^2}\,dx\qquad(2)$$

$$=\log(x^2+1)+\tan^{-1}x-\frac{1}{2(x^2+1)}+\int\frac{dx}{(x^2+1)^2}\cdot\qquad(3)$$

By example 23 of § 183 we have

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x. \tag{4}$$

From (3) and (4) we obtain (1).

From (2) we see that to resolve the given fraction into partial fractions by undetermined coefficients we should assume

$$\frac{2x^3+x^2+3x+2}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}.$$

Note. The solution of the following examples should be deferred until after reading Chapter V.

EXAMPLES.

1. $\displaystyle\int \frac{2x\,dx}{(1+x)(1+x^2)^2} = \frac{1}{4}\log\frac{x^2+1}{(x+1)^2} + \frac{1}{2}\frac{x-1}{x^2+1}.$

2. $\displaystyle\int \frac{dx}{x(x^2+1)^3} = \frac{1}{2}\log\frac{x^2}{1+x^2} + \frac{1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2}.$

3. $\displaystyle\int \frac{x^3+x-1}{(x^2+2)^2}\,dx = \frac{1}{2}\log(x^2+2) + \frac{2-x}{4(x^2+2)} - \frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}}.$

4. $\displaystyle\int \frac{x^5-2x+1}{x^2(x^2+1)^2}\,dx = \log\frac{(x^2+1)^{3/2}}{x^2} - \frac{3x^2+x+2}{2x(x^2+1)} - \frac{3}{2}\tan^{-1}x.$

5. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+1)^2}$

$$= \frac{1}{4}\log\frac{x^2+1}{(x-1)^2} - \frac{1}{4(x^2+1)} - \frac{1}{4(x-1)} + \frac{1}{4}\tan^{-1}x.$$

6. $\displaystyle\int \frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2} = \frac{3x^2-x}{(x-1)(x^2+1)} + \log\frac{(x-1)^2}{x^2+1} + \tan^{-1}x.$

CHAPTER IV.

INTEGRATION BY RATIONALIZATION.

174. Rationalization by substitution. Chapters I and III provide for the integration of any *rational* algebraic differential whether it is entire or fractional in form.

In some *irrational* differentials we can substitute a new variable so related to the old that the new differential will be rational and therefore integrable.

175. *A differential containing no surd except a linear base, as* a + bx, *affected with fractional exponents, can be rationalized by assuming* $a + bx = z^n$, *where* n *is the lowest common denominator of the several fractional exponents.*

For if $a + bx = z^n$, x, dx, and the fractional powers of $a + bx$ will each be rational in terms of z.

Hence, the new function in z will be rational.

Ex. 1. $$\int \frac{dx}{(x-2)^{5/6} + (x-2)^{2/3}} = 6(x-2)^{1/6} - 6\log[(x-2)^{1/6} + 1].$$

Here the linear base is $x - 2$, and $n = 6$; hence, we assume

$$x - 2 = z^6.$$

$$\therefore dx = 6z^5\,dz,\ (x-2)^{5/6} = z^5,\ (x-2)^{2/3} = z^4.$$

$$\therefore \int \frac{dx}{(x-2)^{5/6} + (x-2)^{2/3}} = \int \frac{6z^5\,dz}{z^5 + z^4} = 6\int \frac{z\,dz}{z+1}$$

$$= 6[z - \log(z+1)]$$

$$= 6(x-2)^{1/6} - 6\log[(x-2)^{1/6} + 1].$$

Ex. 2. $\displaystyle\int\frac{x^{1/2}-x^{2/3}}{2\,x^{1/6}}\,dx=\frac{3}{8}x^{4/3}-\frac{x^{3/2}}{3}.$

Here the linear base is x, and $n=6$; hence, we assume

$$x=z^6.$$

$$\therefore\int\frac{x^{1/2}-x^{2/3}}{2\,x^{1/6}}\,dx=3\int(z^7-z^8)\,dz$$

$$=\tfrac{3}{8}z^8-\tfrac{1}{3}z^9=\tfrac{3}{8}x^{4/3}-\tfrac{1}{3}x^{3/2}.$$

EXAMPLES.

1. $\displaystyle\int x\sqrt{a+bx}\,dx=-\frac{2\,(2\,a-3\,bx)\,(a+bx)^{3/2}}{15\,b^2}.$

2. $\displaystyle\int\frac{x\,dx}{\sqrt{a+bx}}=-\frac{2\,(2\,a-bx)}{3\,b^2}\sqrt{a+bx}.$

3. $\displaystyle\int_0^x\frac{dx}{(1+x)^{3/2}+(1+x)^{1/2}}=2\tan^{-1}\sqrt{1+x}-\frac{\pi}{2}.$

4. $\displaystyle\int_a^x\frac{x\,dx}{(a+bx)^{3/2}}=\frac{2\,(2\,a+bx)}{b^2\sqrt{a+bx}}-\frac{2\,(2+b)\sqrt{a}}{b^2\sqrt{1+b}}.$

5. $\displaystyle\int\frac{x^2\,dx}{(1+x)^{2/3}}=3\,(1+x)^{1/3}\left[\frac{(1+x)^2}{7}+\frac{1-x}{2}\right].$

6. $\displaystyle\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}$, when $a>0$,

$\displaystyle\qquad=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}$, when $a<0$.

7. $\displaystyle\int\frac{x^{1/3}\,dx}{x^{1/2}+2\,x^{2/3}}$

$\displaystyle\qquad=\frac{3}{4}x^{2/3}-\frac{1}{2}x^{1/2}+\frac{3}{8}x^{1/3}-\frac{3}{8}x^{1/6}+\frac{3}{16}\log(1+2\,x^{1/6}).$

8. $\displaystyle\int_0^x\frac{dx}{x^{1/2}+x^{1/3}}=2\,x^{1/2}-3\,x^{1/3}+6\,x^{1/6}-6\log(x^{1/6}+1).$

9. $\displaystyle\int\frac{x^{1/6}+1}{x^{7/6}+x^{5/4}}\,dx=-\frac{6}{x^{1/6}}+\frac{12}{x^{1/12}}+2\log x-24\log(x^{1/12}+1).$

10. $\displaystyle\int\frac{xdx}{(2\,x+2)^{3/4}+4\,(2\,x+2)^{1/4}}$

$\displaystyle=(2/5)\,(x+36)\,(2\,x+2)^{1/4}-(4/3)\,(2\,x+2)^{3/4}-28\tan^{-1}\left(\frac{x+1}{8}\right)^{1/4}.$

176. *A differential containing no surd except* $\sqrt{x^2 + ax + b}$ *can be rationalized by assuming* $\sqrt{x^2 + ax + b} = z - x$.

For if
$$\sqrt{x^2 + ax + b} = z - x,$$
$$ax + b = z^2 - 2zx,$$
$$x = \frac{z^2 - b}{2z + a}, \quad dx = \frac{2(z^2 + az + b)}{(2z + a)^2} dz,$$
$$\sqrt{x^2 + ax + b} = z - x = \frac{z^2 + az + b}{2z + a}.$$

Hence, x, dx, and $\sqrt{x^2 + ax + b}$ are each expressed rationally in terms of z.

Ex. $$\int \frac{dx}{x\sqrt{x^2 + x + 1}} = \log \frac{\sqrt{x^2 + x + 1} + x - 1}{\sqrt{x^2 + x + 1} + x + 1}. \qquad (1)$$

Assume
$$\sqrt{x^2 + x + 1} = z - x; \qquad (2)$$
then
$$x + 1 = z^2 - 2xz,$$
$$x = \frac{z^2 - 1}{2z + 1}, \quad dx = \frac{2(z^2 + z + 1)\,dz}{(2z + 1)^2}.$$
$$\sqrt{x^2 + x + 1} = z - x = \frac{z^2 + z + 1}{2z + 1}.$$
$$\therefore \int \frac{dx}{x\sqrt{x^2 + x + 1}} = 2\int \frac{dz}{z^2 - 1} = \log \frac{z - 1}{z + 1}.$$

Substituting for z its value in (2), we obtain (1).

177. *A differential containing no surd except* $\sqrt{-x^2 + ax + b}$ *can be rationalized by assuming*
$$\sqrt{-x^2 + ax + b} = (\beta - x)z,$$
where $\beta - x$ *is one of the factors of* $-x^2 + ax + b$.

Denoting the other factor of $-x^2 + ax + b$ by $x + \gamma$, assume
$$\sqrt{-x^2 + ax + b} = \sqrt{(\beta - x)(x + \gamma)} = (\beta - x)z; \qquad (1)$$
then
$$x + \gamma = (\beta - x)z^2, \quad x = \frac{\beta z^2 - \gamma}{z^2 + 1}.$$

Hence, x and therefore dx and $\sqrt{-x^2 + ax + b}$ are expressed rationally in terms of z.

NOTE. In § 176 the coefficient of x^2 is $+1$; in § 177 it is -1.

EXAMPLES.

1. $$\int \frac{dx}{x\sqrt{x^2+2x-1}} = 2\tan^{-1}(x+\sqrt{x^2+2x-1}).$$

2. $$\int \frac{dx}{x\sqrt{x^2-x+2}} = \frac{\sqrt{2}}{2}\log\frac{\sqrt{x^2-x+2}+x-\sqrt{2}}{\sqrt{x^2-x+2}+x+\sqrt{2}}.$$

3. $$\int \frac{\sqrt{x^2+2x}}{x^2}\,dx = \log(x+1+\sqrt{x^2+2x}) - \frac{4}{x+\sqrt{x^2+2x}}.$$

4. $$\int \frac{dx}{(2+3x)\sqrt{4-x^2}} = \frac{\sqrt{2}}{8}\log\frac{\sqrt{4+2x}-\sqrt{2-x}}{\sqrt{4+2x}+\sqrt{2-x}}.$$

Assume $\sqrt{4-x^2} = \sqrt{(2-x)(2+x)} = (2-x)z$;

then $2+x = (2-x)z^2$,

$$x = \frac{2z^2-2}{z^2+1}, \qquad dx = \frac{8z\,dz}{(z^2+1)^2},$$

$$\sqrt{4-x^2} = (2-x)z = \frac{4z}{z^2+1},$$

$$2+3x = \frac{8z^2-4}{z^2+1}.$$

$$\therefore \int \frac{dx}{(2+3x)\sqrt{4-x^2}} = \frac{1}{2}\int\frac{dz}{2z^2-1} = \frac{\sqrt{2}}{8}\log\frac{\sqrt{2}z-1}{\sqrt{2}z+1}$$

$$= \frac{\sqrt{2}}{8}\log\frac{\sqrt{4+2x}-\sqrt{2-x}}{\sqrt{4+2x}+\sqrt{2-x}}.$$

5. $$\int_2^x \frac{dx}{x\sqrt{2+x-x^2}} = \frac{\sqrt{2}}{2}\log\frac{\sqrt{2+2x}-\sqrt{2-x}}{\sqrt{2+2x}+\sqrt{2-x}}.$$

6. $$\int_a^x \frac{\sqrt{ax-x^2}}{x^2}\,dx = 2\tan^{-1}\sqrt{\frac{a-x}{x}} - 2\sqrt{\frac{a-x}{x}}.$$

7. $$\int_0^x \frac{dx}{(1+x)\sqrt{2+x-x^2}} = -\frac{2}{3}\left(\frac{2-x}{1+x}\right)^{1/2} + \frac{2\sqrt{2}}{3}$$

8. $$\int_1^x \frac{dx}{(1+x)\sqrt{x^2+x+1}} = \log\left(\frac{x+\sqrt{x^2+x+1}}{2+x+\sqrt{x^2+x+1}}\cdot\frac{3+\sqrt{3}}{1+\sqrt{3}}\right).$$

9. $$\int_0^x \frac{x\,dx}{(3+2x-x^2)^{3/2}} = \frac{3+x}{4\sqrt{3+2x-x^2}} - \frac{\sqrt{3}}{4}.$$

Assume $\sqrt{3+2x-x^2} = (3-x)z$.

178. *Irrational differentials of the form* $\mathrm{x^m (a + bx^n)^{r/s} dx}$, *where* r *and* s *are integers and* s *is positive, can be rationalized by assuming,*

I. $\mathrm{a + bx^n = z^s}$, *when* $\dfrac{\mathrm{m+1}}{\mathrm{n}}$ *is an integer or zero.*

II. $\mathrm{a + bx^n = z^s x^n}$, *when* $\dfrac{\mathrm{m+1}}{\mathrm{n}} + \dfrac{\mathrm{r}}{\mathrm{s}}$ *is an integer or zero.*

Assume, $$a + bx^n = z^s;$$

then $$(a + bx^n)^{r/s} = z^r, \tag{1}$$

$$x = \left(\frac{z^s - a}{b}\right)^{1/n}, \qquad x^m = \left(\frac{z^s - a}{b}\right)^{m/n}; \tag{2}$$

$$dx = \frac{s}{bn} z^{s-1} \left(\frac{z^s - a}{b}\right)^{\frac{1}{n}-1} dz. \tag{3}$$

Multiplying (1), (2), and (3) together, we obtain

$$x^m (a + bx^n)^{r/s} dx = \frac{s}{bn} z^{r+s-1} \left(\frac{z^s - a}{b}\right)^{\frac{m+1}{n}-1} dz. \tag{4}$$

The second member of (4) is rational, and therefore integrable, when $(m + 1)/n$ is an integer or zero; hence I.

Assume $$a + bx^n = z^s x^n, \text{ or } x^n = a(z^s - b)^{-1};$$

then $$(a + bx^n)^{r/s} = (z^s x^n)^{r/s} = z^r a^{r/s} (z^s - b)^{-r/s}, \tag{1}$$

$$x = a^{1/n} (z^s - b)^{-1/n}, \qquad x^m = a^{m/n} (z^s - b)^{-m/n}, \tag{2}$$

$$dx = -\frac{s}{n} a^{1/n} z^{s-1} (z^s - b)^{-\frac{1}{n}-1} dz. \tag{3}$$

Multiplying (1), (2), and (3) together, we obtain

$$x^m (a + bx^n)^{\frac{r}{s}} dx = -\frac{s}{n} a^{\frac{m+1}{n}+\frac{r}{s}} (z^s - b)^{-\left(\frac{m+1}{n}+\frac{r}{s}+1\right)} z^{r+s-1} dz.$$

The last member is rational, and therefore integrable, when $\dfrac{m+1}{n} + \dfrac{r}{s}$ is an integer or zero; hence II.

EXAMPLES.

1. $$\int x^5 (a+bx^3)^{-1/2}dx = \frac{2}{9b^2}(a+bx^3)^{3/2} - \frac{2a}{3b^2}\sqrt{a+bx^3}.$$

Here $\frac{m+1}{n} = \frac{5+1}{3} = 2$, and $s = 2$; hence, we assume

$$a + bx^3 = z^2; \quad \therefore (a+bx^3)^{-1/2} = z^{-1}, \tag{1}$$

$$x^6 = \frac{(z^2-a)^2}{b^2}; \quad \therefore 6x^5dx = \frac{4}{b^2}(z^2-a)\,z\,dz. \tag{2}$$

Multiplying (1) by (2) and dividing by 6, we obtain

$$\int \frac{x^5dx}{(a+bx^3)^{1/2}} = \frac{2}{3b^2}\int (z^2-a)\,dz = \frac{2}{9b^2}z^3 - \frac{2a}{3b^2}z$$

$$= \frac{2}{9b^2}(a+bx^3)^{3/2} - \frac{2a}{3b^2}(a+bx^3)^{1/2}.$$

2. $$\int \frac{dx}{x\sqrt{a^2+x^2}} = \frac{1}{a}\log\frac{x}{\sqrt{a^2+x^2}+a}.$$

Here $\frac{m+1}{n} = \frac{-1+1}{2} = 0$, and $s = 2$; hence, we assume

$$a^2 + x^2 = z^2;$$

$$\therefore x\,dx = z\,dz, \quad x^2 = z^2 - a^2.$$

$$\therefore \int \frac{dx}{x\sqrt{a^2+x^2}} = \int \frac{dz}{z^2-a^2} = \frac{1}{2a}\log\frac{z-a}{z+a}$$

$$= \frac{1}{2a}\log\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$$

$$= \frac{1}{a}\log\frac{x}{\sqrt{a^2+x^2}+a}.$$

3. $$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\log\frac{x}{\sqrt{a^2-x^2}+a}.$$

4. $$\int_0^x \frac{x^3dx}{(2-3x^2)^{3/2}} = \frac{4-3x^2}{9\sqrt{2-3x^2}} - \frac{4}{9\sqrt{2}}.$$

5. $$\int_0^x \frac{x^3dx}{\sqrt{a+bx^2}} = \frac{bx^2-2a}{3b^2}\sqrt{a+bx^2} + \frac{2a^{3/2}}{3b^2}.$$

6. $\displaystyle\int \frac{dx}{x^4\sqrt{1+x^2}} = \frac{(2x^2-1)\sqrt{1+x^2}}{3x^3}.$

Here $\dfrac{m+1}{n} + \dfrac{r}{s} = \dfrac{-4+1}{2} - \dfrac{1}{2} = -2$, and $s = 2$; hence, we assume

$$1 + x^2 = z^2x^2;$$

$$\therefore x = (z^2-1)^{-1/2}, \quad x^{-4} = (z^2-1)^2, \tag{1}$$

$$dx = -(z^2-1)^{-3/2}z\,dz, \tag{2}$$

$$(1+x^2)^{-1/2} = z^{-1}x^{-1} = z^{-1}(z^2-1)^{1/2}. \tag{3}$$

Multiplying (1), (2), and (3) together, we obtain

$$\int \frac{dx}{x^4(1+x^2)^{1/2}} = -\int (z^2-1)\,dz = z - \frac{z^3}{3},$$

where $\qquad z = \sqrt{1+x^2}/x.$

7. $\displaystyle\int_0^x \frac{a\,dx}{(1+x^2)^{3/2}} = \frac{ax}{\sqrt{1+x^2}}.$

8. $\displaystyle\int_0^x \frac{dx}{(1+x^2)^{5/2}} = \frac{x(2x^2+3)}{3(1+x^2)^{3/2}}.$

9. $\displaystyle\int_0^x \frac{x^2\,dx}{(a+bx^2)^{5/2}} = \frac{x^3}{3a(a+bx^2)^{3/2}}.$

10. $\displaystyle\int \frac{dx}{x^2(a+bx^2)^{3/2}} = -\frac{a+2bx^2}{a^2x(a+bx^2)^{1/2}}.$

11. $\displaystyle\int \frac{dx}{x^2(a+x^3)^{5/3}} = -\frac{3x^3+2a}{2a^2x(a+x^3)^{2/3}}.$

CHAPTER V.

INTEGRATION BY PARTS. REDUCTION FORMULAS.

179. Integration by parts. By differentiation we have

$$d\,(uv) \equiv udv + vdu.$$

Integrating both members and transposing, we obtain

$$\int udv \equiv uv - \int vdu. \tag{1}$$

The use of formula (1) is called *integration by parts.*

In applying (1) to particular examples the factors u and dv should be so chosen that dv is directly integrable and vdu is a known form or one easier to integrate than udv.

Ex. $\int x \log x\, dx = \frac{x^2}{2} \log x - \frac{x^2}{4}\cdot$

Let
$$u = \log x\,;\ \text{then } dv = x\,dx,$$
$$du = dx/x, \qquad v = x^2/2.$$

Substituting in (1), we obtain

$$\int \log x \cdot x\,dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} \tag{2}$$
$$= (x^2/2) \log x - x^2/4.$$

In (1) the second product can be obtained from the first by integrating its second factor, and the third product from the second by differentiating its first factor.

The following examples will illustrate how we can by the use of this law abbreviate the applications of (1).

Ex. 1. $\int x \cdot \cos x dx = x \cdot \sin x - \int \sin x dx$

$$= x \sin x + \cos x.$$

Ex. 2. $\int x \cdot e^{ax} dx = x \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot dx$

$$= e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$$

Ex. 3. $\int x^3 (a - x^2)^{1/2} dx = \int x^2 \cdot (a - x^2)^{1/2} x dx$

$$= x^2 \;[-\frac{1}{3}(a - x^2)^{3/2}] + \int \frac{1}{3}(a - x^2)^{3/2} \cdot 2\, x dx$$

$$= -\frac{x^2}{3}(a - x^2)^{3/2} - \frac{2}{15}(a - x^2)^{5/2}.$$

EXAMPLES.

1. $\int \log x dx = x (\log x - 1).$

2. $\int x^n \log x dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right);$

and $\int_0^1 x^n \log x dx = -\frac{1}{(n+1)^2}.$ § 86, example 6

3. $\int_0^\pi x \sin x dx = \left[-x \cos x + \sin x\right]_0^\pi = \pi.$

4. $\int_0^1 \sin^{-1} x dx = \left[x \sin^{-1} x + \sqrt{1 - x^2}\right]_0^1 = \frac{\pi}{2} - 1.$

5. $\int_0^1 \tan^{-1} x dx = \left[x \tan^{-1} x - \frac{1}{2}\log(1 + x^2)\right]_0^1 = \frac{\pi}{4} - \frac{\log 2}{2}.$

6. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1 - x^2}.$

7. $\int \cot^{-1} x dx = x \cot^{-1} x + \frac{1}{2}\log(1 + x^2).$

8. $$\int \sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}. \tag{1}$$

$$\int \sqrt{a^2-x^2}\cdot dx = \sqrt{a^2-x^2}\cdot x + \int \frac{x^2dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} - \int \sqrt{a^2-x^2}dx.$$

Transposing the last term and dividing by 2, we obtain the result in (1).

9. $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right).$$

10. $$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right).$$

11. $$\int x^2\sin x dx = 2x\sin x - (x^2-2)\cos x. \tag{1}$$

$$\int x^2\cdot\sin x\,dx = x^2\cdot(-\cos x) + 2\int \cos x\cdot x\,dx \tag{2}$$

$$\int x\cdot\cos x\,dx = x\cdot\sin x - \int \sin x dx$$

$$= x\sin x + \cos x. \tag{3}$$

Substituting in (2) the value in (3), we obtain (1).

12. $$\int x^m\sin x\,dx = -x^m\cos x + m\int x^{m-1}\cos x\,dx.$$

13. $$\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$$

14. $$\int x^m\cos x\,dx = x^m\sin x - m\int x^{m-1}\sin x\,dx.$$

15. $$\int \operatorname{versin}^{-1}x\,dx = (x-1)\operatorname{versin}^{-1}x + \sqrt{2x-x^2}.$$

16. $$\int xe^{cx}\,dx = \frac{e^{cx}}{c^2}(cx-1).$$

17. $\displaystyle\int x^2 e^{cx}\,dx = \frac{e^{cx}}{c}\left(x^2 - \frac{2x}{c} + \frac{2}{c^2}\right).$

18. $\displaystyle\int x^m e^{cx}\,dx = \frac{x^m e^{cx}}{c} - \frac{m}{c}\int x^{m-1} e^{cx}\,dx.$

19. $\displaystyle\int x^3 e^{cx}\,dx = \frac{e^{cx}}{c}\left(x^3 - \frac{3\cdot x^2}{c} + \frac{3\cdot 2\cdot x}{c^2} - \frac{3\cdot 2\cdot 1}{c^3}\right).$

20. $\displaystyle\int x^3 a^x\,dx = \frac{a^x}{\log a}\left[x^3 - \frac{3\cdot x^2}{\log a} + \frac{3\cdot 2\cdot x}{(\log a)^2} - \frac{3\cdot 2\cdot 1}{(\log a)^3}\right].$

21. $\displaystyle\int \frac{e^{cx}\,dx}{x^m} = -\frac{1}{m-1}\,\frac{e^{cx}}{x^{m-1}} + \frac{c}{m-1}\int \frac{e^{cx}}{x^{m-1}}\,dx.$

22. $\displaystyle\int e^{cx}\log x\,dx = \frac{e^{cx}\log x}{c} - \frac{1}{c}\int \frac{e^{cx}}{x}\,dx.$

23. $\displaystyle\int x^n(\log x)^2\,dx = \frac{x^{n+1}}{n+1}\left[(\log x)^2 - \frac{2}{n+1}\log x + \frac{2}{(n+1)^2}\right].$

24. $\displaystyle\int x\cos^{-1}x\,dx = \frac{x^2}{2}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x.$

25. $\displaystyle\int x^2\sin^{-1}x\,dx = \frac{x^3}{3}\sin^{-1}x + \frac{1}{9}(x^2+2)\sqrt{1-x^2}.$

26. $\displaystyle\int \frac{x^2 dx}{1+x^2}\tan^{-1}x = \left(x - \frac{1}{2}\tan^{-1}x\right)\tan^{-1}x - \log\sqrt{1+x^2}.$

180. Additional standard formulas. For convenience of reference we write below the formulas obtained from examples 8, 9, and 10 of § 179, and examples 2 and 3 of § 178.

$$\int\sqrt{a^2-u^2}\,du \equiv \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C. \tag{1}$$

$$\int\sqrt{u^2\pm a^2}\,du$$

$$\equiv \frac{u}{2}\sqrt{u^2\pm a^2} \pm \frac{a^2}{2}\log\left(u+\sqrt{u^2\pm a^2}\right) + C. \tag{2}$$

$$\int\frac{du}{u\sqrt{a^2\pm u^2}} \equiv \frac{1}{a}\log\frac{u}{a+\sqrt{a^2\pm u^2}} + C. \tag{3}$$

181. Reduction formulas. A formula by which any integral not directly obtainable is made to depend upon a standard form or on a form easier to integrate than the original function is called a *reduction formula.* Thus, the formula for integration by parts is a general reduction formula.

Many special formulas are obtained by applying this general formula to particular forms.

182. Reduction formulas for $\int x^m (a + bx^n)^p dx$.

$$\int x^m (a + bx^n)^p dx$$

$$\equiv \frac{x^{m-n+1}(a + bx^n)^{p+1}}{b(np + m + 1)} - \frac{a(m - n + 1)}{b(np + m + 1)} \int x^{m-n}(a + bx^n)^p dx, \tag{A}$$

$$\text{or } \frac{x^{m+1}(a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1} \int x^m (a + bx^n)^{p-1} dx, \tag{B}$$

$$\text{or } \frac{x^{m+1}(a + bx^n)^{p+1}}{a(m + 1)} - \frac{b(np + m + n + 1)}{a(m + 1)} \int x^{m+n}(a + bx^n)^p dx, \tag{C}$$

$$\text{or } -\frac{x^{m+1}(a + bx^n)^{p+1}}{an(p + 1)} + \frac{np + m + n + 1}{an(p + 1)} \int x^m (a + bx^n)^{p+1} dx. \tag{D}$$

Formula (A) decreases m by n.
Formula (B) decreases p by 1.
Formula (C) increases m by n.
Formula (D) increases p by 1.
Formulas (A) and (B) fail when $np + m + 1 = 0$.
Formula (C) fails when $m + 1 = 0$.
Formula (D) fails when $p + 1 = 0$.

When (A), (B), or (C) fails, the method of § 178 is applicable. When (D) fails, $p = -1$ and previous methods apply.

183. Proof of formulas (A), (B), (C), **and** (D).

Integrating by parts when $u = x^{m-n+1}$, we have

$$\int x^m (a + bx^n)^p dx$$

$$= \frac{x^{m-n+1}(a + bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)} \int x^{m-n} (a + bx^n)^{p+1} dx. \quad (1)$$

$$\int x^{m-n} (a + bx^n)^{p+1} dx = \int x^{m-n} (a + bx^n)^p (a + bx^n)\, dx$$

$$= a \int x^{m-n} (a + bx^n)^p dx + b \int x^m (a + bx^n)^p dx. \quad (2)$$

Substituting the last member of (2) for its equal in (1), and solving for $\int x^m (a + bx^n)^p dx$, we obtain (A).

Solving (A) for $\int x^{m-n} (a + bx^n)^p dx$, and substituting in the resulting identity $m + n$ for m, we obtain (C).

$$\int x^m (a + bx^n)^p dx = \int x^m (a + bx^n)(a + bx^n)^{p-1} dx$$

$$= a \int x^m (a + bx^n)^{p-1} dx + b \int x^{m+n} (a + bx^n)^{p-1} dx. \quad (3)$$

Substituting, in (A), $m + n$ for m and $p - 1$ for p, we obtain

$$\int x^{m+n} (a + bx^n)^{p-1} dx$$

$$= \frac{x^{m+1}(a + bx^n)^p}{b(np + m + 1)} - \frac{a(m+1)}{b(np + m + 1)} \int x^m (a + bx^n)^{p-1} dx. \quad (4)$$

Substituting in (3), and combining similar terms, we obtain (B).

Solving (B) for $\int x^m (a + bx^n)^{p-1} dx$, and substituting $p + 1$ for p, we obtain (D).

EXAMPLES.

1. $$\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Ze

Here $m = 2$, $n = 2$, $p = -1/2$, $a = a^2$, $b = -1$.
Substituting these values in (A), we obtain

$$\int x^2(a^2-x^2)^{-1/2}dx = \frac{x(a^2-x^2)^{1/2}}{-2} - \frac{a^2}{-2}\int\frac{dx}{\sqrt{a^2-x^2}}$$
$$= -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

We use formula (A), because decreasing m by n in the given differential reduces it to a known form.

2. $$\int\frac{x^4dx}{\sqrt{a^2-x^2}} = -\left(\frac{x^3}{4}+\frac{3a^2x}{4\cdot 2}\right)\sqrt{a^2-x^2}+\frac{3a^4}{4\cdot 2}\sin^{-1}\frac{x}{a}.$$

3. $$\int\frac{x^2dx}{\sqrt{x^2\pm a^2}} = \frac{x}{2}\sqrt{x^2\pm a^2}\mp\frac{a^2}{2}\log(x+\sqrt{x^2\pm a^2}).$$

4. $$\int\frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{x}{\sqrt{a^2-x^2}+a}.$$

Apply (C), and then use (3) of § 180.

5. $$\int\frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2}-\frac{1}{2a^3}\log\frac{x}{a+\sqrt{x^2+a^2}}.$$

6. $$\int\frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$$

7. $$\int\frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}.$$

8. $$\int_a^x\frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$$

9. $$\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$$

Apply (A), and then use (1) of § 180.

10. $$\int x^2\sqrt{x^2 \pm a^2}\,dx$$
$$= \frac{x}{8}(2\,x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

11. $$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\sec^{-1}\frac{x}{a}.$$

Apply (B).

12. $$\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} + a\log\frac{x}{a + \sqrt{x^2 + a^2}}.$$

13. $$\int_a^x \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} + a\log\frac{x}{a + \sqrt{a^2 - x^2}}.$$

14. $$\int (a^2 - x^2)^{3/2}\,dx = \frac{x}{8}(5\,a^2 - 2\,x^2)\sqrt{a^2 - x^2} + \frac{3\,a^4}{8}\sin^{-1}\frac{x}{a}.$$

15. $$\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$$

Apply (C), and then use (1) of § 180.

16. $$\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\,dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

17. $$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

Apply (D).

18. $$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$$

19. $$\int \frac{x^2dx}{(a^2 - x^2)^{3/2}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$$

20. $$\int \frac{x^2dx}{(x^2 \pm a^2)^{3/2}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

21. $$\int (x^2 \pm a^2)^{3/2}\,dx$$
$$= \frac{x}{8}(2\,x^2 \pm 5\,a^2)\sqrt{x^2 \pm a^2} + \frac{3}{8}a^4\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

22. $$\int \frac{dx}{(x^2 + a^2)^{5/2}} = \frac{x(3a^2 + 2x^2)}{3a^4(x^2 + a^2)^{3/2}}.$$

23. $$\int \frac{dx}{(x^2 + a^2)^2} = \frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^3} \tan^{-1}\frac{x}{a}.$$

24. $$\int \frac{dx}{(x^2 + a^2)^3} = \frac{x}{4a^2(x^2 + a^2)^2} + \frac{3x}{8a^4(a^2 + x^2)} + \frac{3}{8a^5} \tan^{-1}\frac{x}{a}.$$

25. $$\int \frac{dx}{(x^2 + a^2)^n}$$
$$= \frac{1}{2(n-1)a^2} \frac{x}{(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}}.$$

26. $$\int \sqrt{2ax - x^2}\, dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x-a}{a}.$$

Reduce the expression to the form in (1), § 180.

27. $$\int x^m \sqrt{2ax - x^2}\, dx = \int x^{m+1/2}\sqrt{2a - x}\, dx$$
$$= -\frac{x^{m-1}(2ax - x^2)^{3/2}}{m+2} + \frac{(2m+1)a}{m+2} \int x^{m-1}\sqrt{2ax - x^2}\, dx.$$

28. $$\int \frac{x^m\, dx}{\sqrt{2ax - x^2}} = -\frac{x^{m-1}\sqrt{2ax - x^2}}{m} + \frac{(2m-1)a}{m} \int \frac{x^{m-1}\, dx}{\sqrt{2ax - x^2}}.$$

29. $$\int \frac{x\, dx}{\sqrt{2ax - x^2}}.$$

30. $$\int \frac{x^2\, dx}{\sqrt{2ax - x^2}}.$$

31. $$\int \frac{dx}{x^m\sqrt{2ax - x^2}}$$
$$= -\frac{\sqrt{2ax - x^2}}{(2m-1)ax^m} + \frac{m-1}{(2m-1)a} \int \frac{dx}{x^{m-1}\sqrt{2ax - x^2}}.$$

CHAPTER VI.

INTEGRATION OF TRIGONOMETRIC FORMS.

184. $\int \tan^n u\,du$ **or** $\int \cot^n u\,du$, **n any integer.**

When n is a positive integer

$$\int \tan^n u\,du = \int \tan^{n-2} u\,(\sec^2 u - 1)\,du$$

$$= \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du.$$

By repeating this process the integration of $\tan^n u\,du$ is made ultimately to depend upon the integration of $\tan u\,du$ or du according as n is odd or even.

When n is a negative integer, as $-m$, we have

$$\tan^{-m} u\,du = \cot^m u\,du,$$

which may be integrated by a process similar to that used above.

Ex. $\int \tan^{-4}\frac{x}{3}dx = \int \cot^4\frac{x}{3}dx = \int \cot^2\frac{x}{3}\left(\csc^2\frac{x}{3} - 1\right)dx$

$$= -3\int \cot^2\frac{x}{3}\,d\cot\frac{x}{3} - \int \cot^2\frac{x}{3}dx$$

$$= -\cot^3\frac{x}{3} - \int\left(\csc^2\frac{x}{3} - 1\right)dx$$

$$= -\cot^3(x/3) + 3\cot(x/3) + x.$$

185. $\int \sec^n u\,du$ or $\int \csc^n u\,du$, n even and positive.

$$\int \sec^n u\,du = \int \sec^{n-2} u \cdot \sec^2 u\,du$$

$$= \int (\tan^2 u + 1)^{(n-2)/2} \cdot d \tan u, \qquad (1)$$

which can be expanded and integrated directly when n is even and positive.

Ex. $\int \sec^6 x\,dx = \int (\tan^2 x + 1)^2 \cdot d \tan x$

$$= \tan^5 x/5 + 2 \tan^3 x/3 + \tan x.$$

186. $\int \tan^m u \sec^n u\,du$ or $\int \cot^m u \csc^n u\,du$, m odd and positive or n even and positive.

$$\int \tan^m u \sec^n u\,du = \int (\sec^2 u - 1)^{(m-1)/2} \sec^{n-1} u \cdot d \sec u, \qquad (1)$$

$$\text{or} \int \tan^m u\,(\tan^2 u + 1)^{(n-2)/2} \cdot d \tan u. \qquad (2)$$

The form in (1) can be expanded and integrated when m is odd and positive; the form in (2) can be when n is even and positive. Compare (2) with (1) of § 185.

Ex. 1. $\int \tan^3 x \sec^5 x\,dx = \int (\sec^2 x - 1) \sec^4 x \cdot d \sec x$

$$= \sec^7 x/7 - \sec^5 x/5.$$

Ex. 2. $\int \cot^{5/2} x \csc^4 x\,dx = -\int \cot^{5/2} x\,(\cot^2 x + 1) \cdot d \cot x$

$$= -\frac{2}{11} \cot^{11/2} x - \frac{2}{7} \cot^{7/2} x.$$

EXAMPLES.

1. $\int \tan^3 x\,dx = \frac{\tan^2 x}{2} + \log \cos x.$

2. $\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \tan x + x.$

3. $\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x.$

4. $\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x.$

5. $\int \sec^8 2x\,dx = \frac{\tan^7 2x}{14} + \frac{3\tan^5 2x}{10} + \frac{\tan^3 2x}{2} + \frac{\tan 2x}{2}.$

6. $\int \csc^6 \frac{x}{2}\,dx = -\frac{2}{5}\cot^5\frac{x}{2} - \frac{4}{3}\cot^3\frac{x}{2} - 2\cot\frac{x}{2}.$

7. $\int \tan^3 x \sec^{-5} x\,dx = -\sec^{-3} x/3 + \sec^{-5} x/5$

$= \cos^5 x/5 - \cos^3 x/3.$

8. $\int \tan^5 x \sec^{5/3} x\,dx = \frac{3}{17}\sec^{17/3} x - \frac{6}{11}\sec^{11/3} x + \frac{3}{5}\sec^{5/3} x.$

9. $\int \tan^{7/2} x \sec^4 x\,dx = \frac{2}{13}\tan^{13/2} x + \frac{2}{9}\tan^{9/2} x.$

10. $\int \frac{\sec^6 x\,dx}{\tan^4 x} = \tan x - 2\cot x - \frac{\cot^3 x}{3}.$

11. $\int \cot^5 x \csc^4 x\,dx = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8}.$

12. $\int \cot^3 x \csc^{7/2} x\,dx.$ 13. $\int \cot^{-3/2} x \csc^4 x\,dx.$

14. $\int (\sec x + \tan x)^4\,dx = \frac{8}{3}(\sec^3 x + \tan^3 x) - 4\sec x + x.$

15. $\int (\tan x + \cot x)^3\,dx = \frac{1}{2}(\tan^2 x - \cot^2 x) + 2\log\tan x.$

187. $\int \sin^m u \cos^n u\,du$, m or n odd and positive.

$$\int \sin^m u \cos^n u\,du = \int \sin^m u\,(1 - \sin^2 u)^{\frac{n-1}{2}}\,d\sin u, \qquad (1)$$

or

$$-\int \cos^n u\,(1 - \cos^2 u)^{\frac{m-1}{2}}\,d\cos u. \qquad (2)$$

The form in (1) can be expanded and integrated when n is odd and positive; the form in (2) can be when m is odd and positive.

Ex. 1. $$\int \sin^{3/5} x \cos^3 x \, dx = \int \sin^{3/5} x \, (1 - \sin^2 x) \, d \sin x$$
$$= \frac{5}{8} \sin^{8/5} x - \frac{5}{18} \sin^{18/5} x.$$

Ex. 2. $$\int \frac{\cos^3 x \, dx}{\sin^4 x} = \int \frac{(1 - \sin^2 x) \, d \sin x}{\sin^4 x} = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}.$$

188. $\int \sin^m u \cos^n u \, du$, m + n even and negative.

$$\int \sin^m u \cos^n u \, du = \int \frac{\sin^m u}{\cos^m u} \cos^{m+n} u \, du$$
$$= \int \tan^m u \sec^{-(m+n)} u \, du,$$

which is directly integrable by the method of § 186 when $m + n$ is even and negative.

Ex. 1. $$\int \frac{\cos^2 x}{\sin^6 x} dx = \int \cot^2 x \csc^4 x \, dx$$
$$= - \int \cot^2 x \, (\cot^2 x + 1) \, d \cot x$$
$$= - \frac{\cot^5 x}{5} - \frac{\cot^3 x}{3}.$$

Ex. 2. $$\int \cos^{-3/5} x \sin^{-7/5} x \, dx = \int \cot^{-3/5} x \cdot \csc^2 x \, dx$$
$$= - \frac{5}{2} \cot^{2/5} x.$$

Ex. 3. $$\int \frac{dx}{\sin^4 x} = \int \tan^{-4} x \, (\tan^2 x + 1) \, d \tan x$$
$$= - \cot x - \frac{\cot^3 x}{3}.$$

EXAMPLES.

1. $\int \sin^5 x\,dx = -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x.$

2. $\int \sin^5 x \cos^3 x\,dx = \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x.$

3. $\int \sin^4 x \cos^3 x\,dx.$

4. $\int \cos^4 x \sin^3 x\,dx.$

5. $\int \cos^{-2} x \sin^3 x\,dx.$

6. $\int \cos^3 x \sin^{-4} x\,dx.$

7. $\int \frac{\sin^2 x}{\cos^6 x}dx = \int \tan^2 x \sec^4 x\,dx = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3}.$

8. $\int \sin^5 x \sqrt[3]{\cos x}\,dx = -3\sqrt[3]{\cos x}\left(\frac{\cos x}{4} - \frac{\cos^3 x}{5} + \frac{\cos^5 x}{16}\right).$

9. $\int \frac{dx}{\sin^{7/3} x \cos^{5/3} x}$

$$= \int \tan^{-7/3} x \sec^4 x\,dx = \frac{3}{2}\tan^{2/3} x - \frac{3}{4}\cot^{4/3} x.$$

10. $\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{\tan^3 x}{3} + 2\tan x - \cot x.$

11. $\int \frac{dx}{\sin^3 x \cos^5 x} = \frac{\tan^4 x}{4} + \frac{3\tan^2 x}{2} - \frac{\cot^2 x}{2} + 3\log\tan x.$

12. $\int \frac{\sin^{3/2} x\,dx}{\cos^{7/2} x} = \frac{2\tan^{5/2} x}{5}.$

13. $\int \frac{\cos^{2/3} x\,dx}{\sin^{8/3} x}.$

189. $\int \sin^m u \cos^n u\,du$, by multiple angles. When m and n are positive integers, $\sin^m u \cos^n u\,du$ can be expressed, by means of trigonometric formulas, in a series of the first degree in the sines and cosines of multiples of u.

Each term of any such series can be integrated directly.

Ex. 1. $\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{x}{2} + \frac{\sin 2x}{4}.$

Ex. 2. $$\int \cos^4 x\,dx = \frac{1}{4}\int (1 + \cos 2x)^2\,dx$$
$$= \frac{1}{4}\int [1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]\,dx$$
$$= 3x/8 + \sin 2x/4 + \sin 4x/32.$$

We shall use this method only when that of § 187 fails; that is, when m and n are both even.

In any such case the trigonometric formulas for $\sin^2 u$, $\cos^2 u$, and $\sin u \cos u$ in terms of $\sin 2u$ and $\cos 2u$ will enable us to transform all terms with even exponents so that they can be integrated by previous methods.

EXAMPLES.

1. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4}.$

2. $\int \sin^4 x\,dx = \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right).$

3. $\int \sin^6 x\,dx = \frac{1}{16}\left(5x - 4\sin 2x + \frac{\sin^3 2x}{3} + \frac{3\sin 4x}{4}\right).$

4. $\int \sin^2 x \cos^4 x\,dx = \frac{1}{16}\left(\frac{\sin^3 2x}{3} + x - \frac{\sin 4x}{4}\right).$

$$\sin^2 x \cos^4 x = (\sin x \cos x)^2 \cos^2 x = \sin^2 2x\,(1 + \cos 2x)/8$$
$$= \sin^2 2x \cdot \cos 2x/8 + (1 - \cos 4x)/16.$$

5. $\int \sin^4 x \cos^2 x\,dx = \frac{1}{16}\left(-\frac{\sin^3 2x}{3} + x - \frac{\sin 4x}{4}\right).$

6. $\int \sin^2 x \cos^2 x\,dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right).$

7. $\int \sin^4 x \cos^4 x\,dx = \frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right).$

190. Reduction formulas for $\int \sin^m x \cos^n x\, dx$.

$$\int \sin^m x \cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx, \qquad (1)$$

or

$$\frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx, \qquad (2)$$

$$\int \frac{\cos^n x\, dx}{\sin^m x} = -\frac{\cos^{n+1} x}{(m-1)\sin^{m-1} x} + \frac{m-n-2}{m-1}\int \frac{\cos^n x\, dx}{\sin^{m-2} x}. \qquad (3)$$

$$\int \frac{\sin^m x\, dx}{\cos^n x} = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} + \frac{n-m-2}{n-1}\int \frac{\sin^m x\, dx}{\cos^{n-2} x}. \qquad (4)$$

191. Proof of formulas in § 190.

Letting $u = \sin^{m-1} x$, and integrating by parts, we have

$$\int \sin^m x \cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\, dx.$$

$$\sin^{m-2} x \cos^{n+2} x = \sin^{m-2} x \cos^n x\,(1 - \sin^2 x);$$

$$\therefore \int \sin^{m-2} x \cos^{n+2} x\, dx$$

$$= \int \sin^{m-2} x \cos^n x\, dx - \int \sin^m x \cos^n x\, dx.$$

Substituting this value in the first identity, and solving for $\int \sin^m x \cos^n x\, dx$, we obtain (1).

Letting $u = \cos^{n-1} x$, and proceeding in a similar manner, we obtain (2).

Substituting $2 - m$ for m in (1), and solving for $\int \frac{\cos^n x\, dx}{\sin^m x}$, we obtain (3).

Substituting $2 - n$ for n in (2), and solving for $\int \frac{\sin^m x\, dx}{\cos^n x}$, we obtain (4).

EXAMPLES.

1. $$\int \sin^m x\,dx = -\frac{\sin^{m-1}x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2}x\,dx.$$

Putting 0 for n in (1) of § 190, we obtain the formula above.

2. By the formula in example 1 show that

(a) $$\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{8}(x - \sin x\cos x).$$

(b) $$\int \sin^6 x\,dx = -\frac{\cos x}{2}\left(\frac{\sin^5 x}{3} + \frac{5}{12}\sin^3 x + \frac{5}{8}\sin x\right) + \frac{5x}{16}.$$

3. $$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

4. $$\int \cos^4 x\,dx = \frac{\sin x}{4}\left(\cos^3 x + \frac{3}{2}\cos x\right) + \frac{3x}{8}.$$

5. $$\int \frac{dx}{\sin^m x} = -\frac{\cos x}{(m-1)\sin^{m-1}x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2}x}.$$

6. $$\int \csc^5 x\,dx = -\frac{\cos x}{4}\left(\frac{1}{\sin^4 x} + \frac{3}{2\sin^2 x}\right) + \frac{3}{8}\log\tan\frac{x}{2}.$$

7. $$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}.$$

8. $$\int \sec^7 x\,dx = \frac{\sin x}{2\cos^2 x}\left(\frac{1}{3\cos^4 x} + \frac{5}{12\cos^2 x} + \frac{5}{8}\right) + (5/16)\log(\sec x + \tan x).$$

9. $$\int \sin^2 x\cos^4 x\,dx = \frac{\sin x}{2}\left(-\frac{\cos^5 x}{3} + \frac{\cos^3 x}{12} + \frac{\cos x}{8}\right) + \frac{x}{16}.$$

10. $$\int \frac{\sin^4 x}{\cos^2 x}dx = \frac{\sin^5 x}{\cos x} + \sin^3 x\cos x - \frac{3}{2}(x - \sin x\cos x)$$
$$= \frac{\sin^3 x}{\cos x} - \frac{3}{2}(x - \sin x\cos x).$$

11. $$\int \frac{\cos^4 x}{\sin^2 x}dx = -\frac{\cos x}{2\sin x}(3 - \cos^2 x) - \frac{3x}{2}.$$

12. $$\int \frac{dx}{\sin^3 x\cos^2 x} = \frac{1}{\cos x} - \frac{\cos x}{2\sin^2 x} + \frac{3}{2}\log\tan\frac{x}{2}.$$

192. $\displaystyle\int\frac{du}{a+b\cos u}$ **and** $\displaystyle\int\frac{du}{a+b\sin u}$.

$$a+b\cos u = a\left(\sin^2\frac{u}{2}+\cos^2\frac{u}{2}\right)+b\left(\cos^2\frac{u}{2}-\sin^2\frac{u}{2}\right)$$

$$=(a-b)\sin^2(u/2)+(a+b)\cos^2(u/2).$$

$$\therefore \int\frac{du}{a+b\cos u}=\frac{2}{\sqrt{a-b}}\int\frac{\sqrt{a-b}\,d\tan(u/2)}{(a-b)\tan^2(u/2)+(a+b)}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{u}{2}\right); \qquad (1)$$

$$\text{or}\int\frac{du}{a+b\cos u}=\frac{-2}{\sqrt{b-a}}\int\frac{\sqrt{b-a}\,d\tan(u/2)}{(b-a)\tan^2(u/2)-(b+a)}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan(u/2)+\sqrt{b+a}}{\sqrt{b-a}\tan(u/2)-\sqrt{b+a}}. \qquad (2)$$

$$a+b\sin u = a\left(\sin^2\frac{u}{2}+\cos^2\frac{u}{2}\right)+2b\sin\frac{u}{2}\cos\frac{u}{2}.$$

$$\therefore \int\frac{du}{a+b\sin u}=\int\frac{\sec^2(u/2)\,du}{a\tan^2(u/2)+2b\tan(u/2)+a}$$

$$=2\int\frac{a\sec^2(u/2)\,du/2}{[a\tan(u/2)+b]^2+(a^2-b^2)}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan(u/2)+b}{\sqrt{a^2-b^2}}, \qquad (3)$$

$$\text{or}\qquad \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan(u/2)+b-\sqrt{b^2-a^2}}{a\tan(u/2)+b+\sqrt{b^2-a^2}}. \qquad (4)$$

If $a > b$ arithmetically, we use the forms in (1) and (3); if $a < b$ arithmetically, we use the forms in (2) and (4); that is, in each case we use that form of the integral which is real.

EXAMPLES.

1. $\displaystyle\int\frac{dx}{5-3\cos 2x}=\frac{1}{2}\int\frac{d(2x)}{5-3\cos 2x}=\frac{1}{4}\tan^{-1}(2\tan x).$

2. $\displaystyle\int\frac{dx}{5+4\sin 2x}=\frac{1}{3}\tan^{-1}\frac{5\tan x+4}{3}.$

3. $\displaystyle\int\frac{dx}{3+5\cos 3x}=\frac{1}{12}\log\frac{\tan(3x/2)+2}{\tan(3x/2)-2}.$

4. $\displaystyle\int\frac{dx}{4+5\sin 3x}=\frac{1}{9}\log\frac{2\tan(3x/2)+1}{2\tan(3x/2)+4}.$

5. $\displaystyle\int\frac{dx}{3-5\cos 2x}=\frac{1}{8}\log\frac{2\tan x+1}{2\tan x-1}.$

193. Integration of trigonometric forms by substitution. Assume $\sin x = z$; then

$$\cos x=(1-z^2)^{1/2},\qquad dx=(1-z^2)^{-1/2}\,dz.$$

$$\therefore\int\sin^m x\cos^n x\,dx=\int z^m(1-z^2)^{(n-1)/2}\,dz.$$

The last form is integrable for all integral values of m and n, positive or negative.

We might have assumed $\cos x = z$ instead of $\sin x = z$.

This method is applicable to any rational trigonometric differential.

EXAMPLES.

1. $\displaystyle\int\sin^4 x\cos^2 x\,dx=\frac{\cos x}{2}\left(\frac{\sin^5 x}{3}-\frac{\sin^3 x}{12}-\frac{\sin x}{8}\right)+\frac{x}{16}.$ (1)

Let $\sin x = z$; $\therefore \cos^2 x = 1-z^2$, $dx=(1-z^2)^{-1/2}\,dz$.

$$\therefore\int\sin^4 x\cos^2 x\,dx=\int z^4(1-z^2)^{1/2}\,dz.$$

$$=\frac{(1-z^2)^{1/2}}{2}\left(\frac{z^5}{3}-\frac{z^3}{12}-\frac{z}{8}\right)+\frac{\sin^{-1}z}{16}.$$

Substituting $\sin x$ for z, we obtain the integral in (1).

2. $\int \sec^5 x \, dx = \frac{\tan x}{4}\left(\sec^3 x + \frac{3}{2}\sec x\right) + \frac{3}{8}\log(\sec x + \tan x).$

Let $\sec x = z$, or $x = \sec^{-1} z$; $\therefore dx = \frac{dz}{z\sqrt{z^2-1}}\cdot$

$$\therefore \int \sec^5 x \, dx = \int z^4 (z^2-1)^{-1/2} dz$$
$$= \frac{1}{4}(z^2-1)^{1/2}\left(z^3 + \frac{3}{2}z\right) + \frac{3}{8}\log\left(z + \sqrt{z^2-1}\right).$$

Substituting $\sec x$ for z, we obtain the required integral.

3. $\int \frac{\cos^4 x \, dx}{\sin x} = \frac{\cos^3 x}{3} + \cos x + \log \tan \frac{x}{2}\cdot$

Assume $\cos x = z$.

4. $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\log \tan \frac{x}{2}\cdot$

5. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \log \tan \frac{x}{2}\cdot$

6. $\int \frac{dx}{\sin x \cos^4 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin x \cos^4 x}$

$$= \int \frac{\sin x \, dx}{\cos^4 x} + \int \frac{dx}{\sin x \cos^2 x}.$$

7. $\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\cos x} - \frac{\cos x}{2\sin^2 x} + \frac{3}{2}\log \tan \frac{x}{2}\cdot$

8. $\int \frac{d\theta}{a + b\tan\theta} = \frac{a\theta}{a^2+b^2} + \frac{b}{a^2+b^2}\log(a\cos\theta + b\sin\theta).$

Assume $\tan\theta = z$.

9. $\int \frac{d\theta}{\cot^2\theta - 1} = \frac{1}{4}\log\cot\left(\frac{\pi}{4} - \theta\right) - \frac{\theta}{2}\cdot$

194. *To prove* $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$; (1)

and $\int e^{ax} \cos bx\, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2}$. (2)

Integrating $e^{ax} \sin bx\, dx$ by parts, first with $u = \sin bx$ and then with $u = e^{ax}$, we obtain (3) and (4).

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx\, dx, \qquad (3)$$

$$\int e^{ax} \sin bx\, dx = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx\, dx. \qquad (4)$$

Subtracting (3) from (4), we obtain (2). Multiplying (3) by a/b and (4) by b/a, and adding the results, we obtain (1).

EXAMPLES.

1. $\int e^{ax}(\sin ax + \cos ax)\, dx = e^{ax} \sin ax / a$.

2. $\int e^{3x}(\sin 2x - \cos 2x)\, dx = e^{3x}(\sin 2x - 5 \cos 2x)/13$.

3. $\int \frac{\sin x}{e^x} dx = -\frac{\sin x + \cos x}{2e^x}$.

4. $\int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}} = z \tan z + \log \cos z$, where $z = \sin^{-1} x$.

$$\int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}} = \int z \cdot \sec^2 z\, dz = z \tan z - \int \tan z\, dz.$$

5. $\int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}$. Put $x = 2z$.

195. Integration by expansion in series. When by any of the preceding methods we cannot integrate a given differential exactly, we can expand the differential in a series, integrate its terms separately, and thus obtain the integral approximately between the limits of convergency of the series.

EXAMPLES.

1. $$\int \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\cdot\lfloor\underline{3}} + \frac{x^5}{5\cdot\lfloor\underline{5}} - \frac{x^7}{7\cdot\lfloor\underline{7}} + \frac{x^9}{9\cdot\lfloor\underline{9}} - \cdots. \qquad (1)$$

$$\sin x = x - \frac{x^3}{\lfloor\underline{3}} + \frac{x^5}{\lfloor\underline{5}} - \frac{x^7}{\lfloor\underline{7}} + \frac{x^9}{\lfloor\underline{9}} - \cdots. \qquad \S\ 94$$

Multiplying by dx/x and integrating, we obtain (1).

2. $$\int \frac{\cos x}{x}\,dx = \log x - \frac{x^2}{2\cdot\lfloor\underline{2}} + \frac{x^4}{4\cdot\lfloor\underline{4}} - \frac{x^6}{6\cdot\lfloor\underline{6}} + \frac{x^8}{8\cdot\lfloor\underline{8}} - \cdots.$$

3. $$\int \frac{e^{ax}}{x}\,dx = \log x + ax + \frac{a^2x^2}{2\cdot\lfloor\underline{2}} + \frac{a^3x^3}{3\cdot\lfloor\underline{3}} + \frac{a^4x^4}{4\cdot\lfloor\underline{4}} + \cdots.$$

4. $$\int \frac{dx}{\sqrt{1+x^4}} = x - \frac{1\cdot x^5}{2\cdot 5} + \frac{1\cdot 3\cdot x^9}{2\cdot 4\cdot 9} - \frac{1\cdot 3\cdot 5\cdot x^{13}}{2\cdot 4\cdot 6\cdot 13} + \cdots.$$

5. $$\int_0^1 \frac{\log x}{1-x}\,dx = -\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right). \qquad (1)$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots, \text{ when } -1 < x < 1;$$

$$\therefore \int_0^1 \frac{\log x}{1-x}\,dx = \int_0^1 (\log x + x\log x + x^2\log x + \cdots)\,dx. \qquad (2)$$

Integrating each term in (2) by the definite integral given in example 2 of § 179, we obtain (1).

6. By integrating $(1+x^2)^{-1}\,dx$ directly and by series, prove that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \qquad (1)$$

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots;$$

$$\therefore \int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx.$$

7. Prove $\sin^{-1}x = x + \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} + \dfrac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots.$

Integrate $dx/\sqrt{1-x^2}$ both directly and by series.

8. Prove $\log(a+x) = \log a + \dfrac{x}{a} - \dfrac{x^2}{2a^2} + \dfrac{x^3}{3a^3} - \dfrac{x^4}{4a^4} + \cdots.$

Integrate $dx/(a+x)$ both directly and by series.

9. Prove $\log(x+\sqrt{1+x^2}) = x - \dfrac{1\cdot x^3}{2\cdot 3} + \dfrac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} - \dfrac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots.$

CHAPTER VII.

LENGTHS AND AREAS OF CURVES. SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION.

196. Lengths of curves. *Rectangular co-ordinates.* Let s denote the length of the arc whose ends are the points (x_0, y_0) and (x, y); then from $ds^2 = dx^2 + dy^2$, by § 165, we have

$$s = \int_{x_0}^{x} [1 + (dy/dx)^2]^{1/2}\, dx, \tag{1}$$

or

$$s = \int_{y_0}^{y} [(dx/dy)^2 + 1]^{1/2}\, dy, \tag{2}$$

according as we express ds in terms of x or of y.

In any given curve we use that formula which gives the simpler expression to integrate.

EXAMPLES.

1. Find s of the semi-cubical parabola $ay^2 = x^3$.

$$(dy/dx)^2 = 9x/4a;$$

$$\therefore s = \int_{x_0}^{x} \left(1 + \frac{9x}{4a}\right)^{1/2} dx \qquad \text{§ 196, (1)}$$

$$= \frac{8a}{27}\left[\left(1 + \frac{9x}{4a}\right)^{3/2} - \left(1 + \frac{9x_0}{4a}\right)^{3/2}\right]. \tag{1}$$

When (x_0, y_0) is the origin, (1) becomes

$$s = \frac{8a}{27}\left[\left(1 + \frac{9x}{4a}\right)^{3/2} - 1\right].$$

2. Find s of the cycloid $x = r \operatorname{vers}^{-1}(y/r) \mp \sqrt{2ry - y^2}$.

$$(dx/dy)^2 = y/(2r - y);$$

$$\therefore s = \sqrt{2r} \int_{y_0}^{y} (2r - y)^{-1/2}\, dy \qquad \text{§ 196, (2)}$$

$$= 2\sqrt{2r}\,[(2r - y_0)^{1/2} - (2r - y)^{1/2}].$$

Putting $y_0 = 0$ and $y = 2r$ and taking twice the result, we find that the length of one arch is $8r$.

3. Find s of the parabola $y^2 = 4px$, (x_0, y_0) being the origin.

$$Ans.\quad s = \frac{y}{4p}\sqrt{4p^2 + y^2} + p\log\frac{y + \sqrt{4p^2 + y^2}}{2p}.$$

4. Find s of the circle $x^2 + y^2 = r^2$, and the circumference.

$$Ans.\quad r\sin^{-1}(x/r) - r\sin^{-1}(x_0/r);\ 2\pi r.$$

5. Find s of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$, and the length of the curve.

$$Ans.\quad \frac{3}{2}a^{1/3}(x^{2/3} - x_0^{2/3});\ 6a.$$

6. Find s of the ellipse $y^2 = (1 - e^2)(a^2 - x^2)$, and the length of curve, e being the eccentricity.

$$s = \int_{x_0}^{x}(a^2 - e^2x^2)^{1/2}\frac{dx}{\sqrt{a^2 - x^2}}; \tag{1}$$

hence, the length of the elliptic quadrant s_q is

$$s_q = \int_0^a (a^2 - e^2x^2)^{1/2}\frac{dx}{\sqrt{a^2 - x^2}}. \tag{2}$$

The integrals in (1) and (2) cannot be obtained directly, but $(a^2 - e^2x^2)^{1/2}$ can be expanded by the binomial theorem, and the terms of the result can be integrated separately. Thus,

$$\begin{aligned} s_q &= a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} - \frac{e^2}{2a}\int_0^a \frac{x^2\,dx}{\sqrt{a^2 - x^2}} - \frac{e^4}{8a^3}\int_0^a \frac{x^4\,dx}{\sqrt{a^2 - x^2}} - \cdots \\ &= \frac{\pi a}{2}\left(1 - \frac{e^2}{2^2} - \frac{3\cdot e^4}{2^2\cdot 4^2} - \frac{3^2\cdot 5\cdot e^6}{2^2\cdot 4^2\cdot 6^2} - \cdots\right). \end{aligned}$$

7. Find s of the ellipse when given by the equations,

$$x = a\sin\theta,\quad y = b\cos\theta,$$

where θ denotes the complement of the eccentric angle.

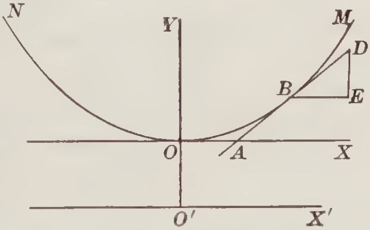
$$\begin{aligned} dx^2 &= a^2\cos^2\theta\,d\theta^2,\quad dy^2 = b^2\sin^2\theta\,d\theta^2; \\ \therefore\ ds^2 &= (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta^2 \\ &= a^2(1 - e^2\sin^2\theta)\,d\theta^2. \end{aligned}$$

$$\therefore\ s = a\int_{\theta_0}^{\theta}(1 - e^2\sin^2\theta)^{1/2}\,d\theta;$$

and $$s_q = a\int_0^{\pi/2}\left(1 - \frac{1}{2}e^2\sin^2\theta - \frac{1}{8}e^4\sin^4\theta - \frac{1}{16}e^6\sin^6\theta - \cdots\right)d\theta.$$

197. *To find the length and the equation of the catenary.*

Let NOM be the curve in which a chain or flexible string hangs when suspended from two fixed points M and N; then NOM is a *catenary*.



Let w denote the weight of a unit length of the chain, and s the length of the arc whose ends are the lowest point $(0, 0)$ and the point (x, y), or B; then the load suspended, or the vertical tension, at B is sw. Denote the horizontal tension, which is the same at all points, by aw. Let DA be a tangent at B; then if $c \cdot BD$ represents the total tension of the chain at B, $c \cdot BE$ and $c \cdot ED$ will represent, respectively, its horizontal and its vertical tension at B.

Hence,
$$\frac{dy}{dx} = \frac{c \cdot ED}{c \cdot BE} = \frac{sw}{aw} = \frac{s}{a}; \tag{1}$$

$$\therefore \frac{s}{a} = \frac{\sqrt{ds^2 - dx^2}}{dx}; \ \therefore dx = \frac{a\,ds}{\sqrt{a^2 + s^2}}.$$

$$\therefore x = a\int_0^s \frac{ds}{\sqrt{a^2 + s^2}} = a \log \frac{s + \sqrt{a^2 + s^2}}{a}. \tag{2}$$

Solving (2) for s, we obtain as the length of OB

$$s = \frac{a}{2}\left(e^{x/a} - e^{-x/a}\right). \tag{3}$$

Eliminating s between (1) and (3), we obtain

$$\int_0^y dy = \frac{1}{2}\int_0^x \left(e^{x/a} - e^{-x/a}\right) dx.$$

$$\therefore y + a = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \tag{4}$$

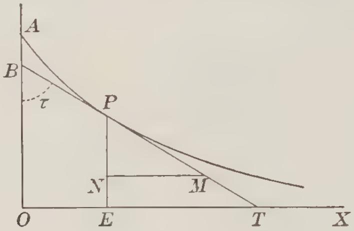
is the equation of the catenary referred to the axes OX and OY.

If $O'O = a$, and the curve be referred to the axes $O'X'$ and $O'Y$, its equation will evidently be

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right). \tag{5}$$

198. *To find the length and the equation of the tractrix.*

The characteristic property of the tractrix is that the length



of its tangent PT is constant.

Denote the constant length of the tangent PT by a.

Let $$PM = ds;$$

then $$-PN = dy, \text{ and } NM = dx.$$

$$\therefore \frac{ds}{dy} = -\frac{PM}{PN} = -\frac{a}{y}. \tag{1}$$

Hence, if s is measured from A, or $(0, a)$, we have

$$s = -a\int_a^y \frac{dy}{y} = a \log \frac{a}{y}. \tag{2}$$

Again, from the figure we have

$$dy/dx = -y/\sqrt{a^2 - y^2}. \tag{3}$$

Integrating (3), remembering that $y = a$ when $x = 0$, we have

$$x = -\sqrt{a^2 - y^2} + a \log\left[(a + \sqrt{a^2 - y^2})/y\right]$$

as the equation of the tractrix.

199. Lengths of polar curves. Let s denote the length of the arc whose ends are (ρ_0, θ_0) and (ρ, θ); then from $ds = \sqrt{\rho^2 d\theta^2 + d\rho^2}$, by § 165, we have

$$s = \int_{\theta_0}^{\theta} \sqrt{\rho^2 d\theta^2 + d\rho^2}, \quad \text{or} \quad s = \int_{\rho_0}^{\rho} \sqrt{\rho^2 d\theta^2 + d\rho^2}, \tag{1}$$

according as ds is expressed in terms of θ or ρ.

$s = a\int (1+\theta^2)^{\frac{1}{2}} d\theta$

EXAMPLES.

1. Find s of the spiral of Archimedes $\rho = a\theta$.

$$ds = a\sqrt{1+\theta^2}\,d\theta;$$

$$\therefore s = a\int_{\theta_0}^{\theta}\sqrt{1+\theta^2}\,d\theta$$

$$= \frac{a}{2}\Big[\theta\sqrt{1+\theta^2} + \log\left(\theta + \sqrt{1+\theta^2}\right)\Big]_{\theta_0}^{\theta}.$$

Putting $\theta_0 = 0$ and $\theta = 2\pi$, we obtain as the length of the first spire

$$a\left[\pi\sqrt{1+4\pi^2} + \frac{1}{2}\log\left(2\pi + \sqrt{1+4\pi^2}\right)\right].$$

2. Find s of the logarithmic spiral $\rho = be^{\theta/a}$.

$$\theta/a = \log(\rho/b) = \log\rho - \log b;$$

$$\therefore \rho d\theta = a d\rho.$$

$$\therefore ds = \sqrt{a^2+1}\,d\rho.$$

$$\therefore s = \sqrt{a^2+1}\int_{\rho_0}^{\rho}d\rho$$

$$= \sqrt{a^2+1}\,(\rho - \rho_0).$$

3. Find s of the cardioid $\rho = 2a(1-\cos\theta)$.

Ans. $s = 8a[\cos(\theta_0/2) - \cos(\theta/2)]$; entire length $= 16a$.

4. The entire length of the curve $\rho = a\sin^3(\theta/3)$ is $3\pi a/2$.

200. Curves in space. Let s denote the length of an arc of a curve in space whose ends are (x_0, y_0, z_0) and (x, y, z), and let Δs denote the length of an infinitesimal arc whose ends are (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$; then

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} + vi^n, \text{ where } n > 1. \qquad § 69$$

$$\therefore ds = \sqrt{dx^2 + dy^2 + dz^2}. \qquad § 71$$

$$\therefore s = \int_{x_0}^{x}\sqrt{dx^2 + dy^2 + dz^2}.$$

EXAMPLES.

1. Find the length of an arc of the helix,

$$x = a\cos\theta,\ y = a\sin\theta,\ z = k\theta.$$

Here $\quad dx = -a\sin\theta d\theta,\ dy = a\cos\theta d\theta,\ dz = kd\theta;$

$$\therefore ds = \sqrt{a^2+k^2}\,d\theta.$$

$$\therefore s = \sqrt{a^2+k^2}\int_{\theta_0}^{\theta} d\theta$$

$$= \sqrt{a^2+k^2}\,(\theta-\theta_0).$$

2. Find s of the curve $y = x^2/2a$, $z = x^3/6a^2$, s being reckoned from the origin.

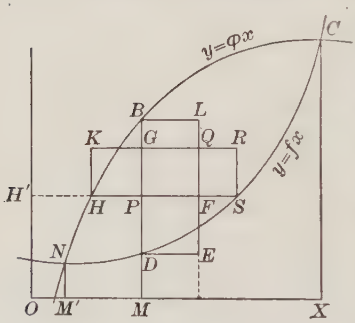
$$s = \int_0^x \left(1+\frac{x^2}{2a^2}\right)dx$$

$$= x + \frac{x^3}{6a^2} = x + z.$$

3. Find s of the curve $y = 2\sqrt{ax} - x$, $z = x - (2/3)\sqrt{x^3/a}$, s being reckoned from the origin.

$$s = \int_0^x \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}} - 1\right)dx = x + y - z.$$

201. Areas of curves. Let NBC be the locus of $y = \phi x$, and $NDSC$ that of $y = fx$. Denote their intersections, N by (x_0, y_0) and C by (x_1, y_1), and the variable area NBD by A.



Let $x = OM$, $dx = DE$; then $dA = DBLE = (\phi x - fx)\,dx$.

$$\therefore A' = \int_{x_0}^{x_1} (\phi x - fx)\,dx, \quad (1)$$

where $\quad A' =$ area $NBCD$.

Writing the equations of the curves in the form

$$x = \phi^{-1}y \text{ and } x = f^{-1}y,$$

in like manner we obtain

$$A' = \int_{y_0}^{y_1} (f^{-1}y - \phi^{-1}y)\, dy. \tag{2}$$

If the locus of $y = fx$ is the x-axis, formula (1) will become identical with (3) in § 166.

EXAMPLES.

1. Find the area bounded by the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.

The parabolas intersect at the points $(0, 0)$ and $(4\,a, 4\,a)$; hence, the limits are $x = 0$ and $x = 4\,a$.

The curves enclosing the area are $y = \sqrt{4\,ax}$ and $y = x^2/4\,a$.

$$\therefore A = \int_0^{4a}\left(\sqrt{4\,ax} - \frac{x^2}{4\,a}\right) dx = \frac{16\,a^2}{3}.$$

2. Find the area bounded by the parabola $x^2 = 4\,ay$ and the witch $y\,(x^2 + 4\,a^2) = 8\,a^3$.

$$\textit{Ans.}\quad A = \int_{-2a}^{2a}\left(\frac{8\,a^3}{x^2 + 4\,a^2} - \frac{x^2}{4\,a}\right) dx = a^2\left(2\,\pi - \frac{4}{3}\right).$$

3. Find the area bounded by the curve $y\,(1 + x^2) = x$, and the right line $y = x/4$. *Ans.* $\log 4 - 3/4$.

4. Find the area of that part of the ellipse $x^2/a^2 + y^2/b^2 = 1$, which lies between the lines $x = x_0$ and $x = x_1$.

Here the bounding curves are

$$y = (b/a)\sqrt{a^2 - x^2}, \qquad y = -(b/a)\sqrt{a^2 - x^2}.$$

$$\therefore A = 2\frac{b}{a}\int_{x_0}^{x_1}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right]_{x_0}^{x_1}. \tag{1}$$

Putting $x_0 = -a$ and $x_1 = a$ in (1), we obtain πab as the area of the given ellipse.

Putting $b = a$ in (1), we obtain the area of that part of the circle $x^2 + y^2 = a^2$ which lies between the lines $x = x_0$, $x = x_1$.

5. Find the area of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, included between the lines $x = x_0$ and $x = x_1$.

$$Ans.\quad \frac{b}{a}\left[x\sqrt{x^2-a^2} - a^2\log\left(x+\sqrt{x^2-a^2}\right)\right]_{x_0}^{x_1}.$$

6. Find the area bounded by the catenary, the x-axis, the y-axis, and any ordinate y.

$$A = \frac{a}{2}\int_0^x \left(e^{x/a} + e^{-x/a}\right) dx = as. \qquad \S\ 197,\ (3)$$

7. Find the area bounded by the tractrix in the first quadrant, the x-axis, and the y-axis.

$$A = \int_0^\infty y\,dx = -\int_a^0 \sqrt{a^2-y^2}\,dy = \pi a^2/4.$$

8. Find the area between the cissoid $y^2(2a - x) = x^3$ and its asymptote. *Ans.* $3\pi a^2$.

9. The area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ is $3\pi a^2/8$.

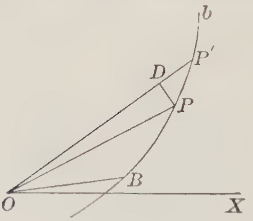
10. The area of one loop of $a^2y^4 = x^4(a^2 - x^2)$ is $4a^2/5$.

11. The area of one loop of $a^4y^2 = b^2x^2(a^2 - x^2)$ is $2ab/3$.

12. Solve the first example by formula (2) in § 201.

13. Find the area between the curve $y^2x = 4a^2(2a - x)$ and its asymptote $x = 0$. *Ans.* $4\pi a^2$.

202. Areas of polar curves. Let B be any fixed point (ρ_0, θ_0) and P any variable point (ρ, θ). Conceive the area BOP as generated by the radius vector ρ, and denote it by A.



With OP as a radius draw arc PD, and let $d\theta = \angle POP'$; then

$$dA = OPD = (\rho^2/2)\,d\theta;$$

$$\therefore A = \frac{1}{2}\int_{\theta_0}^{\theta} \rho^2 d\theta. \qquad (1)$$

For the proof of (1) by limits see § 71, example 11.

EXAMPLES.

1. Find the area of the cardioid $\rho = 2a(1 - \cos\theta)$.

$$\text{Area} = 2a^2 \int_0^{2\pi} (1 - \cos\theta)^2\, d\theta = 6\pi a^2.$$

2. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

$$\text{Area} = 2a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = a^2.$$

3. Find the area between the first and the second spire of the spiral of Archimedes $\rho = a\theta$.

$$\text{Area} = \frac{a^2}{2}\int_{2\pi}^{4\pi} \theta^2\, d\theta - \frac{a^2}{2}\int_0^{2\pi} \theta^2\, d\theta = 8a^2\pi^3.$$

4. The area generated by the radius vector of the logarithmic spiral $\rho = e^{a\theta}$ from $\theta = 0$ to $\theta = \pi/2$ is $(e^{\pi a} - 1)/4a$.

5. The area of one loop of the curve $\rho = a \sin 2\theta$ is $\pi a^2/8$.

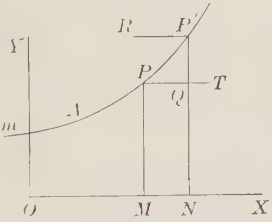
6. The area of one loop of the curve $\rho = a \sin 3\theta$ is $\pi a^2/12$.

7. The area of a sector of the spiral $\rho\theta = a$ is $(\theta - \theta_0)\, a^2/2\,\theta\theta_0$.

8. The area of a sector of the spiral $\rho^2\theta = a^2$ is $\frac{a^2}{2} \log \frac{\theta}{\theta_0}\cdot$

9. The whole area of the curve $\rho = a \cos 2\theta$ is $\pi a^2/2$.

203. Areas of surfaces of revolution. Let A be a fixed point (x_0, y_0) and P a variable point (x, y) on the curve mAP. Let $AP = s$, and $PP' = \Delta s$. Draw PT and $P'R$ each parallel to OX and equal to Δs. Let S denote the surface generated by the revolution of AP about the x-axis; then ΔS equals the surface generated by PP'. Evidently



$$\text{surface } PT < \Delta S < \text{surface } P'R;$$

that is,

$$2\pi y \Delta s < \Delta S < 2\pi(y + \Delta y)\,\Delta s.$$

Let $\Delta s = i;$

then $\Delta S = 2\pi y \Delta s + vi^n$, where $n > 1$.

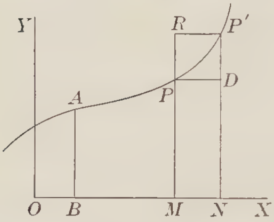
$$\therefore dS = 2\pi y ds; \text{ or } S = 2\pi \int_0^s y ds. \qquad (1)$$

In any particular example, yds is obtained in terms of x, y, or any other variable as may happen to be convenient.

Similarly, if the y-axis is the axis of revolution, we have

$$S = 2\pi \int_0^s x ds. \qquad (2)$$

204. Volumes of solids of revolution. Let A be a fixed point (x_0, y_0) and P a variable point (x, y). Let V denote the volume of the solid generated by revolving $BAPM$ about OX.



Conceive this solid as generated by a circle whose centre moves along OX, and whose variable radius is the ordinate y of the curve AP.

When $x = OM$ let $dx = MN$; then

$$dV = \text{cylinder } MPDN = \pi y^2 dx. \qquad \S\ 11$$

$$\therefore V = \pi \int_{x_0}^x y^2 dx. \qquad (1)$$

Similarly, when the y-axis is the axis of revolution, we have

$$V = \pi \int_{y_0}^y x^2 dy. \qquad (2)$$

The proofs of (1) and (2) by the method of limits are left as exercises for the reader.

In the following examples, a *segment* of a solid of revolution means the portion included between two planes perpendicular to its axis; and a *zone* means the *convex* surface of a segment.

EXAMPLES.

1. Find the area of a zone of a sphere.

Here $yds = rdx.$

$$\therefore S = 2\pi\int_0^s yds = 2\pi r\int_{x_0}^x dx$$

$$= 2\pi r(x - x_0).$$

The entire surface $= 2\Big[2\pi rx\Big]_0^r = 4\pi r^2.$

2. Find the area of a zone of the surface generated by the cycloid revolving about its base.

$$yds = \sqrt{2r}\,(2r - y)^{-1/2}ydy.$$

$$\therefore S = 2\pi\sqrt{2r}\int_{y_0}^y y\,(2r - y)^{-1/2}dy$$

$$= 2\pi\sqrt{2r}\Big[-\frac{2}{3}(4r + y)(2r - y)^{1/2}\Big]_{y_0}^y.$$

The entire surface $= 4\pi\sqrt{2r}\Big[-\frac{2}{3}(4r + y)(2r - y)^{1/2}\Big]_0^{2r}$

$$= 64\pi r^2/3.$$

3. Find the area of a zone of a prolate spheroid.

The generating curve is $y^2 = (1 - e^2)(a^2 - x^2)$ and

$$yds = \frac{b}{a}\sqrt{a^2 - e^2x^2}\,dx.$$

$$\therefore S = 2\pi\frac{b}{a}\int_{x_0}^x\sqrt{a^2 - e^2x^2}\,dx$$

$$= \pi\frac{b}{a}\Big[x\sqrt{a^2 - e^2x^2} + \frac{a^2}{e}\sin^{-1}\frac{ex}{a}\Big]_{x_0}^x.$$

The entire surface $= 2\pi b[b + (a/e)\sin^{-1}e].$

4. Find the area of a zone of the surface generated by the catenary revolving about the x-axis.

Here $yds = \frac{a}{4}(e^{x/a} + e^{-x/a})^2dx.$

$$\therefore S = \pi\Big[\frac{a^2}{4}(e^{2x/a} - e^{-2x/a}) + ax\Big]_{x_0}^x.$$

5. Find the area of a zone of the surface generated by the tractrix revolving about the x-axis. *Ans.* $2\pi a(y_0 - y)$.

6. Find the area of a zone of the paraboloid of revolution.

Ans. $\frac{\pi}{3p}[(4p^2 + y^2)^{3/2} - (4p^2 + y_0^2)^{3/2}]$.

7. The entire surface generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis is $12\pi a^2/5$.

8. The surface generated by revolving the catenary about the y-axis, from $x = 0$ to $x = a$, is $2\pi a^2(1 - e^{-1})$.

9. Find the volume of a segment of the prolate spheroid.

$$V = \pi\int_{x_0}^{x} y^2 dx = \pi\frac{b^2}{a^2}\int_{x_0}^{x}(a^2 - x^2)\,dx$$

$$= \pi\frac{b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_{x_0}^{x}.$$

The entire volume
$$= \pi\frac{b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi ab^2,$$

which is two-thirds of the circumscribed cylinder of revolution.

Putting $b = a$ we obtain the volume of a segment and the entire volume of a sphere whose radius is a.

10. The volume of the oblate spheroid is two-thirds that of the circumscribed cylinder of revolution.

11. The volume of the paraboloid is one-half the circumscribed cylinder of revolution.

12. The volume of the solid generated by revolving an arch of the cycloid about its base is five-eighths of the circumscribed cylinder.

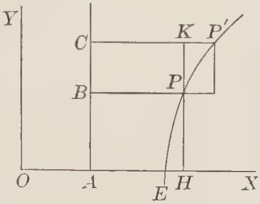
Here
$$V = 2\pi\int_0^{2r}\frac{y^3dy}{\sqrt{2ry - y^2}}.$$

13. Find the volume of the solid generated by the revolution of the tractrix about the x-axis.

$$\text{Volume} = \pi\int_0^{x} y^2dx = -\pi\int_a^0\sqrt{a^2 - y^2}\,y\,dy = \pi a^3/3.$$

14. The entire volume generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis is $32\pi a^3/105$.

15. Find the volume of a segment of the solid generated by the revolution of the curve $f(x, y) = 0$ about the line $x = a$.



Let AB be the line $x = a$, and let P be any point on the curve $f(x, y) = 0$, or EP; then $AH = x - a$.

Let $BC = \Delta y = i$; then

$$\Delta V = \pi (x - a)^2 \Delta y + vi^n,$$

where $n > 1$.

$$\therefore V = \pi \int_{y_0}^{y} (x - a)^2 dy. \qquad (1)$$

The student should prove (1) by the method of § 204.

16. If the figure bounded by $x = a$ and the parabola $y^2 = 4px$ is revolved about the line $x = a$ as an axis, the volume of the solid generated is $32 \pi a^2 \sqrt{pa} / 15$.

17. Find the volume of the solid generated by the revolution of the cissoid about its asymptote. *Ans.* $2 \pi^2 a^3$.

205. Let V denote the volume generated by any plane figure moving parallel to a fixed plane. Let x denote the distance of the generating figure from some fixed point, and let ϕx denote its area; then, evidently,

$$\Delta V \text{ lies between } \phi(x) \cdot \Delta x \text{ and } \phi(x + \Delta x) \cdot \Delta x.$$

Hence, when $\Delta x = i$,

$$\Delta V = \phi(x) \cdot \Delta x + vi^n, \text{ where } n > 1.$$

$$\therefore V = \int \phi(x) \, dx, \qquad (1)$$

the limits being so chosen as to include the volume sought.

EXAMPLES.

1. Find the volume of any pyramid or cone.

Let B denote the area of the base and a the altitude.

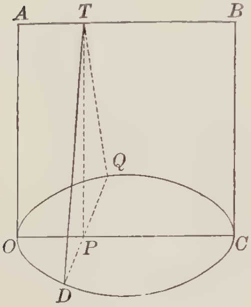
Let ϕx denote the area of a section parallel to the base at the distance x from the vertex. Then by geometry we have

$$\phi x : B = x^2 : a^2; \quad \therefore \phi x = Bx^2 / a^2.$$

Conceive the solid as generated by this variable section moving from the vertex to the base; then by (1) of § 205 we have

$$V = (B/a^2)\int_0^a x^2dx = B \cdot a/3.$$

2. Find the volume of a right conoid with circular base, the radius of the base being r, and the altitude a.



Conceive the solid as generated by the section DTQ moving to the right, and denote OP its perpendicular distance from O by x.

$$OC = AB = 2r,$$
$$OA = CB = a.$$
$$\therefore \phi x = PQ \times PT$$
$$= a\sqrt{2rx - x^2}.$$
$$\therefore V = a\int_0^{2r}\sqrt{2rx - x^2}\,dx$$
$$= \pi r^2 a/2.$$

3. A rectangle moves parallel to and from a fixed plane, one side varying as its distance from this plane, and the other as the cube of this distance. At the distance of 3 feet the rectangle becomes a square of 4 feet. Find the volume then generated. *Ans.* $9\frac{3}{5}$ cubic feet.

4. An isosceles triangle moves perpendicular to the plane of the ellipse $x^2/a^2 + y^2/b^2 = 1$, its base is the double ordinate of the ellipse, and its vertical angle $2A$ is constant. Find the volume generated by the triangle.

Ans. $4ab^2 \cot A/3$.

5. A woodman fells a tree 2 feet in diameter, cutting halfway through from each side. The lower face of each cut is horizontal, and the upper face makes an angle of 45° with the horizontal. How much wood does the man cut out? *Ans.* 4/3 cubic feet.

6. Obtain formula (1) in § 205 by the method of proof employed in § 204.

CHAPTER VIII.

DOUBLE AND TRIPLE INTEGRATION. APPLICATIONS.

206. Double and triple integrals. If we reverse the operations represented by $\frac{\partial^2 u}{dx\,dy}\,dx\,dy$, we obtain the function u.

That is,
$$u = \int\int \frac{\partial^2 u}{dx\,dy}\,dx\,dy, \tag{1}$$
which indicates two successive integrations, the first with reference to y, x and dx being regarded as constants, and the second with reference to x, y being regarded as a constant.

In (1) the right-hand sign of integration is used with the variable y; that is, the signs of integration are taken from right to left in the same order as the differentials.

Let u' denote the definite integral when the limits for x are x_0 and x_1, and those for y are y_0 and y_1; then
$$u' = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\partial^2 u}{dx\,dy}\,dx\,dy. \tag{2}$$

Ex. 1.
$$\begin{aligned}\int_0^a\int_0^b xy\,(x-y)\,dx\,dy &= \int_0^a x\,dx\left[\frac{xy^2}{2} - \frac{y^3}{3}\right]_0^b\\ &= \int_0^a\left(\frac{x^2b^2}{2} - \frac{xb^3}{3}\right)dx\\ &= a^2b^2(a-b)/6.\end{aligned}$$

Oftentimes the limits of the first integration are functions of the variable of the second.

Ex. 2.
$$\int_b^{2b}\int_y^{y^2/b}(x+y)\,dy\,dx = \int_b^{2b}\left(\frac{y^4}{2b^2} + \frac{y^3}{b} - \frac{3y^2}{2}\right)dy = \frac{67b^3}{20}.$$

The second member of (1) denotes what is called an *indefinite double integral,* and the second member of (2) a *definite* *double integral.* Similarly, we have *indefinite* and *definite triple* and *multiple integrals.*

Ex. 3. $$\int_a^{2a}\int_0^x\int_y^x xyz\,dx\,dy\,dz = \int_a^{2a}\int_0^x xy\,dx\,dy\int_y^x zdz$$
$$= \int_a^{2a}\frac{x\,dx}{2}\int_0^x y\,(x^2-y^2)\,dy$$
$$= \int_a^{2a}\frac{x^5}{8}\,dx = \frac{21\,a^6}{16}\cdot$$

EXAMPLES.

1. $$\int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dx\,dy = \frac{a^3}{3}\cdot$$

2. $$\int_0^a\int_0^{\pi/2}\rho^2\sin\phi\,d\rho\,d\phi = \frac{a^3}{3}\cdot$$

3. $$\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dx\,dy = \frac{a^4}{8}\cdot$$

4. $$\int_{b/2}^b\int_0^{\rho/b}\rho\,d\rho\,d\theta = \frac{7\,b^2}{24}\cdot$$

5. $$\int_0^b\int_{y-b}^{2y} xy\,dy\,dx = \frac{11\,b^4}{24}\cdot$$

6. $$\int_0^{\pi/2}\int_{2b\cos\phi}^{2a\cos\phi}\rho\,d\phi\,d\rho = \frac{\pi}{2}(a^2-b^2).$$

7. $$\int_a^b\int_\beta^\gamma\rho^2\sin\theta\,d\rho\,d\theta = \frac{b^3-a^3}{3}(\cos\beta-\cos\gamma).$$

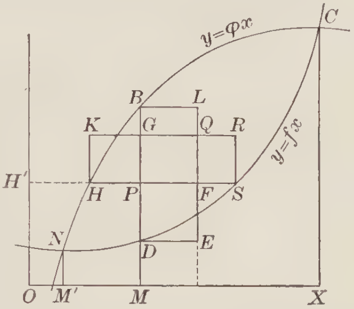
8. $$\int_b^a\int_{-y}^y (x-a)\,(y-b)\,dy\,dx = a^3b - \frac{ab^3+2\,a^4}{3}\cdot$$

9. $$\int_{-a}^a\int_{-a}^a\int_{-a}^a (y^2+z^2)\,k\,dz\,dy\,dx = \frac{16}{3}ka^5.$$

10. $$\int_a^b\int_0^a\int_b^{2b} x^2y^2z\,dx\,dy\,dz = \frac{1}{6}a^3b^2\,(b^3-a^3).$$

11. $$\int_0^2\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz = \frac{e^8-3}{8} - \frac{3\,e^4}{4} + e^2.$$

207. Areas by double integration. *Rectangular co-ordinate.* Let NBC be the locus of $y = \phi x$, and $NDSC$ that of $y = fx$. Denote their intersections, N by (x_0, y_0) and C by (x_1, y_1), and the curvilinear area $NBCD$ by A. Let P be any point (x, y) in this area, x and y being independent.



Let $PF = dx$ and $PG = dy$.

When x and dx are constants, $PGQF$, or $dx\,dy$, will be the differential of the area $DPFE$. Hence, integrating $dx\,dy$ between the limits MD and MB, or fx and ϕx, we obtain the area $DBLE$, or $(\phi x - fx)\,dx$, which is the differential of the area NDB. Integrating $(\phi x - fx)\,dx$ between the limits x_0 and x_1, we obtain the area $NBCD$, or A.

Hence,
$$A = \int_{x_0}^{x_1}\int_{fx}^{\phi x} dx\,dy. \tag{1}$$

When y and dy are constants, $PGQF$, or $dy\,dx$, will be the differential of the area $HPGK$. Hence, integrating $dy\,dx$ between the limits $H'H$ and $H'S$, we obtain the area $HSRK$, which is the differential of the area NHS. Integrating this between the limits $M'N$ and XC, we obtain the area $NBCD$, as before.

Hence,
$$A = \int\int dy\,dx, \tag{2}$$

the limits being taken so as to include the required area.

The order of integration, therefore, is indifferent, provided the limits assigned in each case be such as to include the area sought.

COR. $\partial^2_{xy} A = dx\,dy$ and $\partial^2_{yx} A = dy\,dx$.

Ex. 1. Find the area bounded by the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.

The parabolas intersect at the points (0, 0) and $(4\,a, 4\,a)$.

Hence, if we use formula (1), the constant limits for x will be 0 and $4\,a$, and the variable limits for y will be $x^2/4\,a$ and $\sqrt{4\,ax}$.

That is, $$\text{area} = \int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dx\,dy = \frac{16\,a^2}{3}.$$

Using formula (2), we obtain

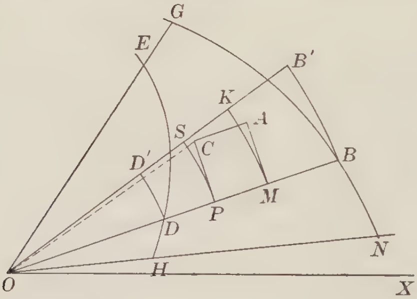
$$\text{area} = \int_0^{4a}\int_{y^2/4a}^{\sqrt{4ay}} dy\,dx = \frac{16\,a^2}{3}.$$

Ex. 2. Find the area between the parabola $y^2 = ax$ and the circle $y^2 = 2\,ax - x^2$.

$$\text{Area} = 2\int_0^{a}\int_{\sqrt{ax}}^{\sqrt{2ax - x^2}} dx\,dy = \frac{\pi a^2}{2} - \frac{4\,a^2}{3}.$$

208. Areas of polar curves by double integration.

Let NBG be the locus of $\rho = \phi\theta$, and HDE that of $\rho = f\theta$.



Let $\angle XON = \theta_0$, and $\angle XOG = \theta_1$.

Denote the area $HEGN$ by A.

Let P be any point (ρ, θ) in this area, ρ and θ being independent.

Let $$PM = d\rho \text{ and } \angle POS = d\theta;$$

then $$\text{arc } PS = \rho d\theta.$$

Construct the rectangle $PCAM$ where $PC = PS = \rho d\theta$; then $$\text{triangle } POC = \text{sector } POS.$$

When θ and $d\theta$ are constant, $PCAM$, or $\rho d\theta d\rho$, will be the differential of the triangle POC, or of its equal, POS. Hence, integrating $\rho d\theta d\rho$ between the limits OD and OB, or $f\theta$ and $\phi\theta$, we obtain the area $DBB'D'$, or $\frac{1}{2}\left[(\phi\theta)^2-(f\theta)^2\right]d\theta$, which is the differential of the area $HDBN$.

Integrating this differential between the limits θ_0 and θ_1, we obtain the area $HEGN$, or A.

Hence,
$$A=\int_{\theta_0}^{\theta_1}\int_{f\theta}^{\phi\theta}\rho\,d\theta\,d\rho. \tag{1}$$

COR. $\partial^2_{\theta\rho}A=\rho d\theta\,d\rho$.

EXAMPLES.

1. Find the area between the two tangent circles $\rho=2a\cos\theta$ and $\rho=2b\cos\theta$, where $a>b$.

$$\text{Area}=2\int_0^{\pi/2}\int_{2b\cos\theta}^{2a\cos\theta}\rho d\theta\,d\rho$$
$$=4(a^2-b^2)\int_0^{\pi/2}\cos^2\theta\,d\theta=\pi(a^2-b^2).$$

2. Find the area, (1) between the first and the second spire of the spiral of Archimedes $\rho=a\theta$; (2) between any two consecutive spires.

3. By double integration find the area, (1) of a rectangle; (2) of a parallelogram; (3) of a triangle.

4. Find the whole area of the curve $(y-mx-c)^2=a^2-x^2$.

Ans. πa^2.

209. Area of any surface by double integration. On the surface $z=f(x, y)$, let P be any point (x, y, z), and Q the point $(x+\Delta x, y+\Delta y, z+\Delta z)$, x and y being independent; then $P'N=\Delta x$ and $P'M=\Delta y$.

Conceive a tangent plane at P, not shown in the figure. The planes through P and Q parallel to the co-ordinate planes XZ and YZ will cut a curved quadrilateral PQ from the surface $z=f(x, y)$, and a parallelogram Pq from the tangent plane.

Let $\quad \Delta x = i$ and $\Delta y = vi$;

then $\quad$ area $PQ =$ area $Pq + vi^n$, where $n > 2$

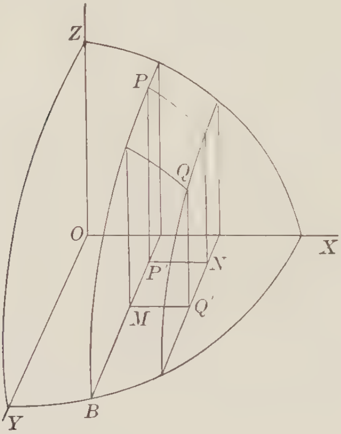
$$= \text{area } P'Q' \cdot \sec\gamma + vi^n, \qquad (1)$$

where γ is the angle which the tangent plane at P makes with the plane XY.

From (1) we have

$$\Delta^2_{xy}S = \Delta x\,\Delta y \sec\gamma + vi^n.$$

$$\therefore \partial^2_{xy}S = \sec\gamma \cdot dx\,dy. \qquad \text{§ 140, Cor.}$$



From analytic geometry we have

$$\sec\gamma = \left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{1/2}.$$

$$\therefore S = \int\!\!\int \left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{1/2} dx\,dy, \qquad (2)$$

the limits being so chosen as to include the required surface.

Let S denote that part of the surface $z = f(x, y)$, z being a one-valued function, which is included by

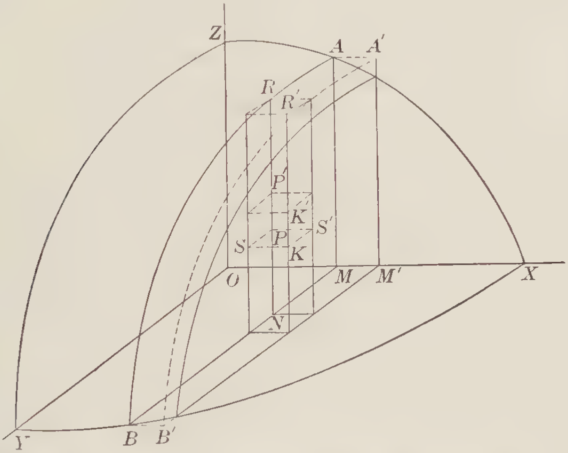
the cylindrical surfaces $y = \phi_0 x$, $y = \phi x$,
and the planes $\quad x = a, \quad x = b$;

then $$S=\int_a^b\int_{\phi_0 x}^{\phi x}\left[1+\left(\frac{\partial z}{dx}\right)^2+\left(\frac{\partial z}{dy}\right)^2\right]^{1/2}dx\,dy. \quad (2)$$

210. Volume of any solid by triple integration.

Let P be any point (x, y, z) within the solid $OZY\text{-}X$, x, y, and z being independent.

Let $$PS' = dx,\ PS = dy,\ PP' = dz.$$



Regarding x, dx, y, and dy as constants, the prism PK', or $dx\,dy\,dz$, will be the differential of the prism NK. Hence, integrating $dx\,dy\,dz$ between the limits $z=0$ and $z=NR$, we obtain $\overline{NR}\,dx\,dy$, or the prism NR', which is the differential of the solid $MM'A'A\text{-}R$.

Integrating $\overline{NR}\,dx\,dy$ between the limits $y=0$ and $y=MB$, we obtain the cylinder $MAB\text{-}B'$, or $\overline{MAB}\,dx$, which is the differential of the solid $OZY\text{-}M$.

Integrating $\overline{MAB}\,dx$ between the limits $x=0$ and $x=OX$, we obtain the volume $OZY\text{-}X$, or V.

Hence, $$V=\iiint dx\,dy\,dz, \quad (1)$$

the limits being so chosen as to include the volume sought.

Let V denote the volume bounded by

the curved surfaces $z = f_0(x, y),\quad z = f(x, y)$;
the cylindrical surfaces $y = \phi_0 x,\quad y = \phi x$;
and the planes $x = a,\quad x = b$;

then
$$V = \int_a^b \int_{\phi_0 x}^{\phi x} \int_{f_0(x, y)}^{f(x, y)} dx\, dy\, dz. \tag{2}$$

COR. $\partial^3_{xyz} V = dx\, dy\, dz,\ \partial^3_{yzx} V = dy\, dz\, dx, \cdots$.

Ex. Find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

The entire volume is eight times that in the first octant, where the limits are

$$z = 0,\quad z = c\sqrt{1 - x^2/a^2 - y^2/b^2};$$
$$y = 0,\quad y = b\sqrt{1 - x^2/a^2};$$
$$x = 0,\quad x = a.$$
$$\therefore V = 8\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} \int_0^{c\sqrt{1 - x^2/a^2 - y^2/b^2}} dx\, dy\, dz = \frac{4\pi abc}{3}.$$

EXAMPLES.

1. Find the volume bounded by the plane $x = a$ and the surface $z^2/c + y^2/b = 2x$.

The entire volume is four times that in the first octant;

$$\therefore V = 4\int_0^a \int_0^{\sqrt{2bx}} \int_0^{\sqrt{c(2x - y^2/b)}} dx\, dy\, dz = \pi a^2 \sqrt{bc}.$$

2. Find the volume bounded by the surfaces,

$$x^2 + y^2 = cz,\ x^2 + y^2 = ax,\ z = 0.$$

Ans. $2\int_0^a \int_0^{\sqrt{ax - x^2}} \int_0^{(x^2 + y^2)/c} dx\, dy\, dz = \frac{3\pi a^4}{32c}$.

3. Find the volume bounded by the cylinder $x^2 + y^2 = r^2$ and the planes $z = 0$ and $z = mx$. *Ans.* $4mr^3/3$.

4. Find the volume bounded by the surface $x^2z^2 + a^2y^2 = c^2x^2$ and the planes $x = 0$ and $x = a$. *Ans.* $\pi c^2 a/2$.

5. Find the area of the zone of the sphere,

$$x^2 + y^2 + z^2 = r^2, \qquad (1)$$

included between the planes $x = a$ and $x = b$.

From (1), $\partial z / dx = -x/z$, $\partial z / dy = -y/z$.

Hence, $\left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{1/2} = \frac{r}{\sqrt{r^2 - x^2 - y^2}}$.

The area required is four times that in the first octant, where the limits are $x = a$, $x = b$, $y = 0$, $y = \sqrt{r^2 - x^2}$;

$$\therefore \text{area} = 4 \int_a^b \int_0^{\sqrt{r^2 - x^2}} \frac{r\,dx\,dy}{\sqrt{r^2 - x^2 - y^2}} = 2\pi r(b - a). \quad \S\ 209,\ (2)$$

6. Find the surface of the cylinder $x^2 + z^2 = r^2$ intercepted by the cylinder $x^2 + y^2 = r^2$. *Ans.* $8r^2$.

211. Solids of revolution. Let P be any point (x, y) in the area $NBCD$ (§ 207, fig.), x and y being independent.

Let $PF = \Delta x$ and $PG = \Delta y$.

Conceive $NBCD$ to revolve through θ radians about OX as an axis; then

$$\theta y \cdot \Delta x\,\Delta y < \Delta_{xy}^2 V < \theta(y + \Delta y) \cdot \Delta x\,\Delta y.$$

Hence, when $\Delta x = i$, and $\Delta y = vi$,

$$\Delta_{xy}^2 V = \theta y\,\Delta x\,\Delta y + vi^n, \text{ where } n > 2.$$

$$\therefore \partial_{xy}^2 V = \theta y\,dx\,dy.$$

$$\therefore V = \theta \int_{x_0}^{x_1} \int_{fx}^{\phi x} y\,dx\,dy. \qquad (1)$$

Putting $\theta = 2\pi$, we obtain the volume generated by a complete revolution of the area.

Cor. If the x-axis cuts the area, formula (1) will give the difference between the volumes generated by the two parts. Hence, $V = 0$ when these two parts generate equal volumes.

212. The **moment of a force about an axis** perpendicular to its line of direction is the product of its magnitude by the perpendicular distance of its line of action from the axis, and measures the tendency of the force to produce rotation about the axis.

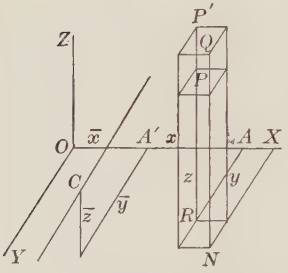
The **force exerted by gravity** on a body varies as the mass of the body, and may be measured by the mass.

The **centre of mass** of a body is a point so situated that the force of gravity produces no tendency in the body to rotate about any axis passing through this point.

The mass of any homogeneous body is the product of its volume by its density.

213. *To find the centre of mass of a body.*

Let the points of the body be referred to the rectangular axes OX, OY, OZ, the plane XY being horizontal. Let m denote the mass of the body, and M the moment of the force of gravity on m about an axis parallel to OY and passing through C, $(\bar{x}, \bar{y}, \bar{z})$.



Let P be any point (x, y, z) in the body, and Q the point

$$(x + \Delta x, y + \Delta y, z + \Delta z).$$

Let Δm equal the mass of the parallelopiped PQ; then

$$(x - \bar{x})\Delta m < \Delta M < (x + \Delta x - \bar{x})\Delta m.$$

Let $\Delta m \doteq \text{point } P;$

then, if $\Delta x = i,\ \Delta m = v_8 i^3,$

and $\Delta M = (x - \bar{x})\Delta m + v i^n,$ where $n > 3$.

$$\therefore dM = (x - \bar{x})\,dm;$$

$$\therefore M = \int x\,dm - \bar{x}\int dm. \tag{1}$$

When $(\bar{x}, \bar{y}, \bar{z})$ is the centre of mass, $M = 0$; hence from (1)

$$\bar{x} = \int x\,dm \bigg/ \int dm. \qquad [1]$$

In like manner we obtain

$$\bar{y} = \frac{\int y\,dm}{\int dm}, \quad \bar{z} = \frac{\int z\,dm}{\int dm}. \qquad [2]$$

To obtain $\bar{z}$ place the z-axis horizontal.

Whether the body is homogeneous or not, dm denotes the mass of a homogeneous solid whose density is that of the body at the point P (x, y, z), (§ 11).

Hence, denoting the volume of dm by dv and the density of the body at P by k, we have $dm = kdv$. Substituting kdv for dm in [1] and [2], we have

$$\bar{x} = \frac{\int xk\,dv}{\int k\,dv}, \quad \bar{y} = \frac{\int yk\,dv}{\int k\,dv}, \quad \bar{z} = \frac{\int zk\,dv}{\int k\,dv}. \qquad [3]$$

When the body is not homogeneous, k is some function of the coördinates of the point (x, y, z), and

$$dv = \partial^3_{xyz} V = dx\,dy\,dz.$$

When the body is homogeneous, k is constant, and formulas [3] become

$$\bar{x} = \frac{\int x\,dv}{\int dv}, \quad \bar{y} = \frac{\int y\,dv}{\int dv}, \quad \bar{z} = \frac{\int z\,dv}{\int dv}. \qquad [4]$$

Cor. In formulas [4] dv may equal $\partial^3_{xyz} V$, $\partial^2 V$, or dV.

For when the body is homogeneous, all points in the plane ARP have the same moment. Hence, to prove [4] we may let Δm equal the mass of the body between the planes ARP and XNQ, or an increment of this mass. In the first case dv will equal dV, and in the second it will equal $\partial^2_{xy} V$ or $\partial^2_{xz} V$.

214. Centre of mass of right cylinders and areas. Let c denote the altitude of the right cylinder whose convex surface is made up of the cylindrical surfaces $y = fx$, $y = \phi x$, and the planes $x = x_0$, $x = x_1$, the plane XY being midway between and parallel to the bases.

Evidently $\bar{z} = 0$.

To find $\bar{x}$ and $\bar{y}$ we have $dv = \partial_{xy}^2 V = c\,dx\,dy$.

Hence, from [4] of § 213, we have

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}, \qquad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}, \tag{5}$$

the limits for x being x_0 and x_1, and for y, fx and ϕx.

As the values of $\bar{x}$ and $\bar{y}$ depend solely on the plane area bounded by the plane curves $y = fx$, $y = \phi x$, and the lines $x = x_0$, $x = x_1$; for convenience the point $(\bar{x}, \bar{y})$ is called the mass-centre of this area.

Cor. 1. If a plane area be revolved about a line through its mass-centre, the two parts will generate equal volumes.

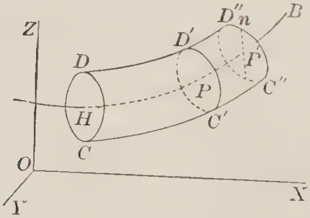
For let this line coincide with the x-axis; then $\bar{y} = 0$, and from [5] we have

$$\theta \int\int y\,dx\,dy = 0. \tag{1}$$

Hence, by Cor. of § 211, the two volumes are equal.

Cor. 2. If an area is symmetrical with respect to the x-axis, $\bar{y} = 0$, and $\bar{x}$ is the same for one of the symmetrical halves as for the whole area.

215. Centre of mass of rods and curves. Suppose the mass-centre of the plane figure CD to move along the curve HPB, its plane being always perpendicular to the curve.



Let A denote the constant area CD; V, the volume of the rod generated by CD; s, the arc HP; and Δs, PP'. Through P' draw in the plane $C''D''$ the line $P'n$ parallel to the plane $C'D'$. On $P'n$ as an axis revolve $C''D''$ until it becomes parallel to $C'D'$.

Then, by Cor. 1 of § 214, we have

$$\Delta V = A \cdot \Delta s + vi^n, \text{ where } n > 1.$$

$$\therefore dV = Ads. \tag{1}$$

Substituting Ads for dv in [4] of § 213, we obtain

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds}, \quad \bar{z} = \frac{\int z\,ds}{\int ds}. \qquad [6]$$

As the values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ depend solely on the curve HB, for convenience $(\bar{x}, \bar{y}, \bar{z})$ is often called the mass-centre of the curve HB.

If the curve is in the plane XY, $\bar{z} = 0$; if in addition it is symmetrical with respect to the x-axis, $\bar{y} = 0$, and $\bar{x}$ is the same for one of the symmetrical halves as for the whole curve.

COR. From (1) we obtain $V = As$. (2)

That is, the volume of the rod CDP equals the area of CD into the length of the arc traced by the mass-centre of CD.

The proofs of equations (1) and (2) fail when CD cuts the evolute of the curve HB.

EXAMPLES.

1. Find the centre of mass of the area bounded by the parabola $y^2 = 4px$ and a double ordinate.

From the symmetry of the curve, $\bar{y} = 0$, and

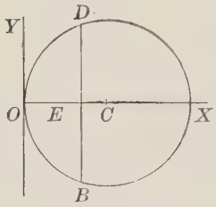
$$\bar{x} = \int_0^x \int_0^{\sqrt{4px}} x\,dx\,dy \Big/ \int_0^x \int_0^{\sqrt{4px}} dx\,dy = 3x/5.$$

2. Find the centre of mass of the area bounded by the semicubical parabola $ay^2 = x^3$ and a double ordinate. *Ans.* $\bar{x} = 5x/7$.

3. Find the centre of mass of the area bounded by the y-axis and the curve $xy^2 = b^2(a - x)$. *Ans.* $\bar{x} = a/4$.

4. Find the centre of mass of the area of the first quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$. *Ans.* $\bar{x} = 4a/3\pi$, $\bar{y} = 4b/3\pi$.

5. Find the centre of mass E of any circular arc BOD.



Let $OB = OD = s$, O being the origin.

The equation of BOD is $y^2 = 2rx - x^2$, r being the radius.

From the symmetry of the curve

$$\bar{y} = 0,$$

and
$$\bar{x} = \frac{\int_0^s x\,ds}{\int_0^s ds} = \frac{r}{s}\int_0^x \frac{x\,dx}{\sqrt{2rx - x^2}}$$

$$= r - ry/s = OE.$$

Hence, $CE = 2ry/2s = r$ chord $BD/$arc BOD.

6. Find the centre of mass of the arc in the first quadrant of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $\bar{x} = \bar{y} = 2a/5$.

7. Find the centre of mass of the arc of a catenary cut off by any horizontal chord.

Ans. $\bar{y} = (ax + ys)/2s$, where $2s$ is the length of the arc.

8. Find the centre of mass of the curve $xy^2 = b^2(a - x)$.

Ans. $\bar{x} = a/4$.

9. The axis of a homogeneous solid of revolution is the x-axis; show that $\bar{y} = \bar{z} = 0$, and

$$\bar{x} = \int\int xy\,dx\,dy \Big/ \int\int y\,dx\,dy. \qquad \S\ 211$$

10. Find the volume of the ring generated by the revolution of an ellipse about an external axis in its own plane, the distance of the centre of the ellipse from the axis being r.

Ans. $2\pi^2 abr$. § 215, Cor.

11. If an arc of a plane curve revolve through θ radians about an external axis in its own plane, the area of the surface generated will be equal to the length of the revolving arc, multiplied by the length of the path described by the mass-centre of this arc.

From [6] of § 215 we obtain

$$\theta\bar{y}\cdot s = \theta\int_0^s y\,ds,\text{ or the theorem.} \qquad \S\ 203$$

12. Find the surface of the ring generated by the revolution of a circle (radius a) about an external axis, the distance of the centre of the circle from the axis being r. *Ans.* $4\pi^2 ar$.

216. The **moment of inertia** of a plane area about a given point in its plane is the limit of the sum of the products obtained by multiplying the area of each infinitesimal portion by the square of its distance from the given point.

Denote by $M.I.$ the moment of inertia of the area $NBCD$, or A, about O (§ 207, fig.). Let P be any point (x, y) in this area, x and y being independent; then

$$\overline{OP}^2 = x^2 + y^2.$$

Let $$PF = \Delta x = i, \text{ and } PG = \Delta y = vi;$$

then $$\Delta^2_{xy}(M.I.) = (x^2 + y^2)\,\Delta x\,\Delta y + vi^n, \text{ where } n > 2.$$

$$\therefore \partial^2_{xy}(M.I.) = (x^2 + y^2)\,dx\,dy.$$

$$\therefore M.I. = \int\int (x^2 + y^2)\,dx\,dy,$$

the limits being taken so as to include the required area.

Ex. 1. Find the moment of inertia about the origin, of the circle

$$x^2 + y^2 = a^2.$$

$$M.I. = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2 + y^2)\,dx\,dy = \frac{\pi a^4}{2}.$$

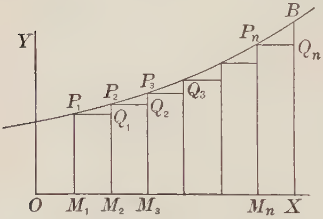
Ex. 2. Find the moment of inertia about the origin, of the smaller area bounded by the x-axis, the parabola $y^2 = 4ax$, and the line $x + y = 3a$.

$$M.I. = \int_{0}^{2a}\int_{y^2/4a}^{3a-y} (x^2 + y^2)\,dy\,dx = \frac{314\,a^4}{35}.$$

CHAPTER IX.

DEFINITE INTEGRAL AS A LIMIT. INTRINSIC EQUATIONS OF CURVES.

217. Definite integral as a limit. Heretofore we have considered differentials as finite; in this article we shall regard them as infinitesimal. A definite integral has been defined as an increment of an indefinite integral. We proceed to show that *a definite integral equals the limit of the sum of an infinite number of infinitesimal differentials.*



To make the theorem and its proof as clear as possible, let us consider the area M_1P_1BX, which we will denote by A.

Let $OM_1 = a$, $OX = b$, and P_1B be the locus of $y = \phi x$.

Divide M_1X into n equal parts, M_1M_2, M_2M_3, $\cdot\cdot\cdot$, M_nX, and divide the area A as in the figure.

Let $\quad dx = M_1M_2 = M_2M_3 = \cdot\cdot\cdot = M_nX,$

then
$$M_1Q_1 = \phi(a)dx, \qquad M_2Q_2 = \phi(a+dx)\,dx,$$
$$M_3Q_3 = \phi(a+2\,dx)\,dx, \cdot\cdot\cdot, \; M_nQ_n = \phi(b-dx)\,dx.$$
$$\therefore A = \phi(a)\,dx + \phi(a+dx)\,dx + \phi(a+2\,dx)\,dx$$
$$+ \cdot\cdot\cdot + \phi(b-dx)\,dx + T, \tag{1}$$

where T is the sum of the triangles

$$P_1Q_1P_2,\; P_2Q_2P_3,\; \cdot\cdot\cdot,\; P_nQ_nB.$$

By the notation of sums, (1) is written

$$A = \sum_a^b \phi(x)\,dx + T.$$

Evidently $T < \overline{XB} \cdot dx$, or $\phi b\, dx$.

Let $n = \infty$; then $dx \doteq 0$; $\therefore T \doteq 0$.

$$\therefore \underset{dx \doteq 0}{\text{limit}} \sum_a^b \phi(x)\, dx = A = \int_a^b \phi(x)\, dx. \tag{2}$$

The first member of (2) denotes the limit of the sum of an infinite number of differentials, each of which is represented by $\phi x\, dx$, x taking in succession the values

$$a,\ a + dx,\ a + 2\,dx,\ \cdots,\ b - dx, \text{ while } dx \doteq 0.$$

When ϕx is constant, $T = 0$, and A equals the sum of the differentials in (1) for all values of dx.

Again consider the volume generated by revolving M_1P_1BX about OX as an axis. Each of the rectangles M_1Q_1, M_2Q_2, $\cdots$, M_nQ_n will generate a cylinder whose volume will be represented by $\pi(\phi x)^2\, dx$. Hence,

$$V = \sum_a^b \pi(\phi x)^2\, dx + T, \text{ where } T < \pi(\phi b)^2\, dx.$$

$$\therefore \underset{dx \doteq 0}{\text{limit}} \sum_a^b \pi(\phi x)^2\, dx = V = \int_a^b \pi(\phi x)^2\, dx. \qquad \S\ 203$$

EXAMPLES.

1. The effect of gravity in making a body tend to rotate about any given axis is the same as if its mass were concentrated at its centre of mass.

From [1] of § 213, by integration we have

$$\bar{x}m = \int_0^m x\, dm = \underset{dm \doteq 0}{\text{limit}} \sum_0^m x\, dm. \tag{1}$$

The given axis being OY, $\bar{x}m$ is what would be the moment of the force of gravity on m if m were concentrated at its centre of mass. The last member of (1) is the limit of the sum of the moments of the force of gravity on all the material points (dm) of m when $dm \doteq 0$. Hence, (1) proves the theorem.

2. Show that the area of a polar curve is the limit of the sum of an infinite number of infinitesimal differentials.

3. Using the figure in § 207 show that

$$\sum_{fx}^{\phi x} dx\,dy = (\phi x - fx)\,dx = \int_{fx}^{\phi x} dx\,dy, \tag{1}$$

and

$$\underset{dx \doteq 0}{\text{limit}} \sum_{x_0}^{x_1} (\phi x - fx)\,dx = A = \int_{x_0}^{x_1} (\phi x - fx)\,dx. \tag{2}$$

(1) holds true whether dy is finite or infinitesimal; for dx being constant the sum is constant.

In (2) the sum varies with dx, and dx must be infinitesimal to cause this sum to approach its limit A.

4. Using the figure in § 210, show that

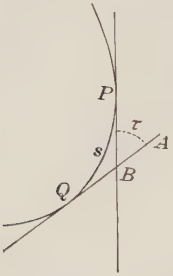
$$\sum_0^{\overline{NR}} dx\,dy\,dz = \overline{NR}\,dx\,dy = \int_0^{\overline{NR}} dx\,dy\,dz; \tag{1}$$

$$\underset{dy \doteq 0}{\text{limit}} \sum_0^{\overline{MB}} \overline{NR}\,dx\,dy = AMB \cdot dx = \int_0^{\overline{MB}} \overline{NR}\,dx\,dy; \tag{2}$$

$$\underset{dx \doteq 0}{\text{limit}} \sum_0^{\overline{OX}} AMB \cdot dx = OZY\text{-}X = \int_0^{\overline{OX}} AMB \cdot dx. \tag{3}$$

In (1) dx and dy are constants, and dz may be either a constant or an infinitesimal. In (2) dx is a constant, but $dy \doteq 0$.

218. Intrinsic equation of a curve. Let s denote the arc between a fixed point, Q, and a variable point, P, of the curve QP, and τ the angle ABP included between the tangents at Q and P; then the equation which expresses the relation between the variables s and τ is called the *intrinsic* equation of the curve.



Ex. 1. Find the intrinsic equation of the circle.

Let QP, or s, be an arc of a circle whose radius is r. Let C denote the centre of this circle; then

$$\tau = \angle ABP = \angle QCP = s/r.$$

Hence, $s = r\tau$ is the intrinsic equation of the circle.

Ex. 2. Find the intrinsic equation of the catenary.

In § 197 let $OB = s$; then $\tau = \angle XAB$;

and $\tan\tau = dy/dx = s/a$. § 197, (1)

Hence, $s = a\tan\tau$

is the intrinsic equation of the catenary.

Ex. 3. Find the intrinsic equation of the tractrix.

In § 198 let $AP = s$; then $\sec\tau = \sec EPT = a/y$.

Hence, $s = a\log\sec\tau$ § 198, (2)

is the intrinsic equation of the tractrix.

219. To obtain the intrinsic equation of a curve from its rectangular or polar equation, we find the values of s and τ and eliminate the other variables between these equations.

Ex. Find the intrinsic equation of the cycloid.

When s is reckoned from the cusp (§ 196, example 2), we have

$$s = 4r(1 - \sqrt{2r-y}/\sqrt{2r})$$

and

$$\cos\tau = dy/ds = \sqrt{2r-y}/\sqrt{2r}.$$

$$\therefore s = 4r(1-\cos\tau). \tag{1}$$

When s is reckoned from the vertex, we have

$$s = -\sqrt{2r}\int_{2r}^{y}(2r-y)^{-1/2}dy = 4r\cdot\sqrt{2r-y}/\sqrt{2r},$$

and

$$\sin\tau = -dy/ds = \sqrt{2r-y}/\sqrt{2r}.$$

$$\therefore s = 4r\sin\tau. \tag{2}$$

220. *If the intrinsic equation of the involute* QP *is* $\mathrm{s} = \mathrm{f}\tau$, *the intrinsic equation of the evolute* $\mathrm{Q_1P_1}$ *is*

$$\mathrm{s} = \mathrm{f}'\tau - \mathrm{f}'0. \tag{1}$$

The curvature of QP is $d\tau/ds$.

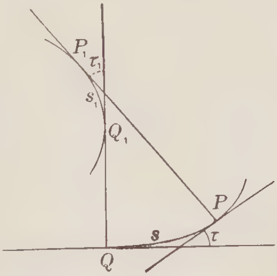
$$\therefore R = ds/d\tau = f'\tau. \tag{2}$$

$$s_1 = P_1P - Q_1Q \qquad \text{§ 119}$$

$$= f'\tau - f'0 \qquad \text{by (2)}$$

$$= f'\tau_1 - f'0, \tag{3}$$

since $\tau_1 = \tau$.



Omitting the subscripts in (3), we have **(1)**.

Ex. 1. The evolute of the tractrix $s = a \log \sec \tau$ is

$$s = a \frac{d(\log \sec \tau)}{d\tau}\Big]_0^\tau = a \tan \tau,$$

which, by example 2 in § 218, is the catenary.

Ex. 2. The evolute of the cycloid $s = 4r(1 - \cos \tau)$ is

$$s = 4r \frac{d(1 - \cos \tau)}{d\tau}\Big]_0^\tau = 4r \sin \tau,$$

which, by the example in § 219, is an equal cycloid.

EXAMPLES.

1. Find the evolute of the catenary $s = a \tan \tau$.

2. Find the intrinsic equation of $x^{2/3} + y^{2/3} = a^{2/3}$ and of its evolute.

Ans. $s = (3a/2) \sin^2 \tau$; $s' = (3a/2) \sin 2\tau$.

3. Find the intrinsic equation of the logarithmic spiral $\rho = be^{\theta/a}$ and of its evolute.

When s is measured from the point $(b, 0)$ where the spiral crosses the initial line, we have

$$s = b\sqrt{1 + a^2}\,(e^{\theta/a} - 1). \qquad \text{§ 199, example 2}$$

Since ψ is constant, $\tau = \theta$.

$$\therefore s = b\sqrt{1 + a^2}\,(e^{\tau/a} - 1).$$

CHAPTER X.

ORDINARY DIFFERENTIAL EQUATIONS.

221. A **differential equation** is an equation which involves one or more differentials or derivatives.

An *ordinary* differential equation is one which involves only one independent variable.

For example,

$$dy = \cos x\, dx, \tag{1}$$

$$d^2y / dx^2 + y = 0, \tag{2}$$

and

$$y = x \cdot dy / dx + r\sqrt{1 + (dy / dx)^2}, \tag{3}$$

are *ordinary* differential equations.

The **order** of a differential equation is the order of the highest differential or derivative which it contains.

The **degree** of a differential equation is that of the highest power to which the highest differential or derivative which it contains is raised, after the equation is freed from fractions and radicals.

Thus equation (1) is of the first order and first degree, (2) is of the second order and first degree, while (3) is of the first order and second degree.

222. The general solution of a differential equation is the most general equation free from differentials or derivatives, from which the former equation may be derived by differentiation.

The general solution of equation (1) in § 221 is

$$y = \sin x + C,$$

where C is the *constant of integration.*

$y = \sin x$, $y = \sin x + 7$, $\cdots$, are particular solutions of (1), which are included in its general solution.

The general solution *may* not include all possible solutions. A solution not included in the general solution is called a *singular solution.* For a discussion of singular solutions the reader will consult some treatise on differential equations.

The general solution of a differential equation of the nth order contains n arbitrary *constants of integration.* It is often called the *complete integral* or *primitive* of the differential equation.

223. In the foregoing chapters it was our object to obtain the general solution of differential equations of the form

$$dy = \phi(x)\,dx.$$

In this chapter we shall extend the process of integration to differential equations of the more general form

$$\mathbf{M}d\mathbf{x} + \mathbf{N}d\mathbf{y} = 0, \qquad (1)$$

where M and N are functions of x and y.

The *variables* in $Mdx + Ndy$ are said to be *separated* when M, or the coefficient of dx, contains x only, and N contains y only.

When $Mdx + Ndy$ is the total differential of some function of x and y, it is called an *exact differential,* and (1) is called an *exact differential equation.*

For example, $x\,dy + y\,dx$ is the exact differential of xy; hence, the general solution of the exact differential equation,

$$x\,dy + y\,dx = 0,$$

is

$$xy = C.$$

224. Equations of the form $\phi_1(\mathbf{x})\,d\mathbf{x} + \phi_2(\mathbf{y})\,d\mathbf{y} = 0.$ (1)

When, as in (1), the variables are separated, a differential equation is solved by integrating its terms separately.

For example, the general solution of the differential equation

$$e^x\,dx + 3y^2\,dy = 0,$$

is

$$e^x + y^3 = C.$$

When an equation is, or may be, written in the form

$$\phi_1(x)\cdot\phi_2(y)\,dx+\phi_3(x)\cdot\phi_4(y)\,dy=0,$$

the variables may be separated by dividing both members by $\phi_2(y)\cdot\phi_3(x)$.

Ex. 1. Solve $(1-x)\,dy-(1+y)\,dx=0.$ (1)

Dividing by $(1-x)(1+y)$ to separate the variables, and integrating, we obtain

$$\log(1+y)+\log(1-x)=\log C. \qquad (2)$$

$$\therefore (1+y)(1-x)=C. \qquad (3)$$

Equations (2) and (3) are two equally correct ways of expressing the general solution of (1).

Solution (3) could be obtained without separating the variables in (1) by noting that $(1-x)\,dy-(1+y)\,dx$ is the exact differential of $(1-x)(1+y)$.

Ex. 2. Solve $(x^2+1)\,dy=(y^2+1)\,dx.$

Here $$\tan^{-1}y=\tan^{-1}x+\tan^{-1}c=\tan^{-1}\frac{x+c}{1-cx}.$$

$$\therefore y=\frac{x+c}{1-cx}.$$

EXAMPLES.

Solve each of the following differential equations:

1. $\dfrac{dy}{dx}=\dfrac{x^2+x+1}{y^2+y+1}.$ $\quad\dfrac{x^3-y^3}{3}+\dfrac{x^2-y^2}{2}+x-y=C.$

2. $\dfrac{x^2+1}{y+1}=xy\dfrac{dy}{dx}.$ $\quad\dfrac{x^2}{2}+\log x=\dfrac{y^3}{3}+\dfrac{y^2}{2}+C.$

3. $(1+x)\,y\,dx+(1-y)\,x\,dy=0.$ $\quad\log(xy)+x-y=C.$

4. $dy=(e^{x-y}+x^2e^{-y})\,dx.$ $\quad 3(e^y-e^x)=x^3+C.$

5. $x\cos^2y\,dx=y\cos^2x\,dy.$
$\quad x\tan x-\log\sec x=y\tan y-\log\sec y+C.$

6. $a(x\,dy+2y\,dx)=xy\,dy.$ $\quad x^2y=Ce^{y/a}.$

7. Find the equation of the family of curves whose slope is $(4x^3+2x+1)/(6y^2+4y)$. $\quad 2y^3+2y^2=x^4+x^2+x+C.$

8. The equation of the family of curves which cross all their radii vectores at the same angle A is $\rho = Ce^{\theta \cot A}$.

Here $d\rho/\rho = \cot A d\theta$; $\therefore \log \rho = \log e^{\theta \cot A} + \log C$.

9. Find the equation of the family of curves

(1) whose slope is $-b^2x/a^2y$;

(2) whose slope is $(e^{x/a} - e^{-x/a})/2$;

(3) whose subtangent is the constant a;

(4) whose subnormal is the constant a;

(5) whose tangent is the constant a.

Ans. $a^2y^2 + b^2x^2 = C$; $y = a(e^{x/a} + e^{-x/a})/2 + C$; $y = Ce^{x/a}$;
$y^2 = 2ax + C$; $x = \sqrt{a^2 - y^2} + a \log[(a - \sqrt{a^2 - y^2})/y] + C$.

10. Find the equation of the curve which passes through the point (a, b), and intersects at right angles each of the series of curves represented by the equation

$$y^2 = 2\alpha x^3,$$

in which α is a variable parameter.

11. Helmholtz's equation for the strength of an electric current, C, at the time t is $C = \frac{E}{R} - \frac{L}{R}\frac{dC}{dt}$, where E, R, and L are given constants. Find the value of C, having given that $C = 0$ when $t = 0$.

Ans. $C = E(1 - e^{-Rt/L})/R$.

225. Equations homogeneous in x and y. After being divided by x^n (n being the degree of each term in x and y), any equation homogeneous in x and y can be put in the form

$$dy = f(y/x)\,dx. \tag{1}$$

Putting $y = vx$ in (1), we obtain

$$v\,dx + x\,dv = f(v)\,dx. \tag{2}$$

The variables in (2) are easily separated; hence, its solution is found by § 224.

Ex. 1. Solve $(x^2 + y^2)\,dx = 2xy\,dy$. (1)

Putting $y = vx$ and dividing by x^2, we obtain

$$(1 + v^2)\,dx = 2v(x\,dv + v\,dx). \tag{2}$$

Separating the variables and integrating, we have

$$\log [x(1 - v^2)] = \log C.$$

Putting y/x for v, the solution becomes

$$x^2 - y^2 = Cx.$$

Ex. 2. Solve $(x^2 + y^2)\,dy = xy\,dx.$ (1)

Putting $y = vx$ and dividing by x^2, we obtain

$$(1 + v^2)(x\,dv + v\,dx) = v\,dx. \qquad (2)$$

Separating the variables and integrating, we have

$$\log x + \log C = v^2/2 - \log v,$$

or

$$Cy = e^{x^2/2y^2}.$$

EXAMPLES.

Solve each of the differential equations:

1. $x^2dy - y^2dx = xy\,dx.$ $\qquad \log x + x/y = C.$

2. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$ $\qquad y = Ce^{-\sqrt{x/y}}.$

3. $y^2dx + x^2dy = xy\,dy.$ $\qquad y = Ce^{y/x}.$

4. $xdy = (y + \sqrt{x^2 + y^2})\,dx.$ $\qquad x^2 = C^2 + 2\,Cy.$

5. $(x + y)\,dy = (y - x)\,dx.$ $\qquad \log(x^2 + y^2) + 2\tan^{-1}(y/x) = C.$

6. $x^2dy = y^2dx.$ $\qquad y - x = Cxy.$

7. $(8\,y + 10\,x)\,dx + (5\,y + 7\,x)\,dy = 0.$ $\qquad (y + x)^2(y + 2\,x)^3 = C.$

8. Find the system of curves at any point of which, as (x, y), the subtangent is equal to the sum of x and y. *Ans.* $y = Ce^{x/y}.$

226. Non-homogeneous equations of the first degree in x and y.

These equations are of the form

$$(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0. \qquad (1)$$

Putting $x' + h$ for x and $y' + k$ for y in (1), we obtain

$$(ax' + by' + ah + bk + c)\,dx + (a'x' + b'y' + a'h + b'k + c')\,dy = 0. \qquad (2)$$

Giving to h and k the values determined by the system,

$$ah + bk + c = 0, \;\; a'h + b'k + c' = 0, \tag{3}$$

equation (2) becomes

$$(ax' + by')\,dx + (a'x' + b'y')\,dy = 0, \tag{4}$$

which is homogeneous in x' and y', and can therefore be solved by the method of § 225.

This method fails when $a'/a = b'/b$; for then h and k in system (3) are infinite or indeterminate.

In this case assume

$$a'/a = b'/b = m, \text{ or } a' = ma, \;\; b' = mb.$$

Equation (1) then assumes the form

$$(ax + by + c)\,dx + [m\,(ax + by) + c']\,dy = 0. \tag{5}$$

Let $ax + by = v$; then $dy = (dv - adx)/b$.

Substituting these values in (5), we obtain

$$[b\,(v + c) - a\,(mv + c')]\,dx + (mv + c')\,dv = 0,$$

where the variables are readily separated.

EXAMPLES.

1. Solve $(2x + 3y - 8)\,dx - (x + y - 3)\,dy = 0.$ (1)

Putting $x' + h$ for x, and $y' + k$ for y, we obtain

$$(2x' + 3y' + 2h + 3k - 8)\,dx' = (x' + y' + h + k - 3)\,dy'. \tag{2}$$

Assume the system

$$2h + 3k - 8 = 0, \;\; h + k - 3 = 0; \quad \text{or} \quad h = 1, \;\; k = 2.$$

Equation (2) then becomes

$$(2x' + 3y')\,dx' = (x' + y')\,dy'. \tag{3}$$

Putting $y' = vx'$, (3) becomes

$$(2 + 3v)\,dx' = (1 + v)\,(vdx' + x'dv),$$

or $$-\frac{dx'}{x'} = \frac{v+1}{(v-1)^2-3}\,dv$$

$$= \left[\frac{v-1}{(v-1)^2-3} + \frac{1}{\sqrt{3}}\left(\frac{1}{v-1-\sqrt{3}} - \frac{1}{v-1+\sqrt{3}}\right)\right] dv.$$

$$\therefore -\log x' = \frac{1}{2}\log\{(v-1)^2-3\} + \frac{1}{\sqrt{3}}\log\frac{v-1-\sqrt{3}}{v-1+\sqrt{3}} + C.$$

where $$x' = x-1 \text{ and } v = \frac{y-2}{x-1}.$$

2. $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0.$

Ans. $(y - x + 1)^2(y + x - 1)^5 = C.$

3. $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0.$

Ans. $x + 2y + \log(2x + y - 1) = C.$

4. $(2y + x + 1)\,dx = (2x + 4y + 3)\,dy.$

Ans. $4x - 8y = \log(4x + 8y + 5) + C.$

5. $(7y + x + 2)\,dx = (3x + 5y + 6)\,dy.$

Ans. $x + 5y + 2 = C(x - y + 2)^4.$

227. Exact differential equations. *The condition that*

$$\mathrm{Mdx} + \mathrm{Ndy} = 0 \qquad (1)$$

may be an exact differential equation is

$$\partial \mathrm{M}/\mathrm{dy} = \partial \mathrm{N}/\mathrm{dx}. \qquad (2)$$

Comparing (1) with the exact differential equation

$$du \equiv \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy = 0, \qquad (3)$$

we obtain

$$M = \partial u/dx, \qquad N = \partial u/dy, \qquad (4)$$

as the conditions that (1) be *exact.*

From conditions (4) by differentiation, we obtain

$$\frac{\partial M}{dy} = \frac{\partial^2 u}{dy\,dx} = \frac{\partial N}{dx}, \text{ or } (2).$$

Condition (2) is called Euler's Criterion of Integrability.

When condition (2) is satisfied, (1) may be solved by regarding y as constant and putting

$$u = \int M dx + fy, \tag{5}$$

and then determining fy so that

$$\partial u / dy = N. \tag{6}$$

Or regarding x as constant, we may put

$$u = \int N dy + fx,$$

and so determine fx that

$$\partial u / dx = M.$$

Equations (5) and (6) involve the conditions in (4).

Ex. Solve $x(x + 2y)\,dx + (x^2 - y^2)\,dy = 0.$ (1)

Here $M = x(x + 2y), \quad N = x^2 - y^2.$ (2)

$$\therefore \partial M / dy = 2x = \partial N / dx;$$

hence, condition (2) of § 227 is fulfilled.

Regarding y as constant, we put

$$u = \int x(x + 2y)\,dx + fy$$

$$= x^3/3 + yx^2 + fy. \tag{3}$$

To determine fy, from (3) and (2) we have

$$\partial u / dy = x^2 + f'y = N = x^2 - y^2.$$

$$\therefore f'y = -y^2, \text{ or } fy = -y^3/3. \tag{4}$$

From (3) and (4) we obtain

$$u = x^3/3 + yx^2 - y^3/3.$$

$$\therefore x^3 + 3yx^2 - y^3 = C$$

is a solution of (1).

EXAMPLES.

Solve each of the following differential equations:

1. $(6xy - y^2)\,dx + (3x^2 - 2xy)\,dy = 0.$ $\quad 3x^2y - y^2x = C.$
2. $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0.$ $\quad x^4 + 6x^2y^2 + y^4 = C.$
3. $(x^2 + y^2)\,dx + 2xy\,dy = 0.$ $\quad x^3 + 3xy^2 = C.$
4. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
   $x^3 - 6x^2y - 6xy^2 + y^3 = C.$
5. $(1 + y^2/x^2)\,dx - (2y/x)\,dy = 0.$ $\quad x^2 - y^2 = Cx.$
6. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0.$ $\quad x^2 + y^2 + 2\tan^{-1}\dfrac{y}{x} = C.$

   In example 6 divide both terms of the fraction by x^2.
7. $e^x(x^2 + y^2 + 2x)\,dx + 2ye^x\,dy = 0.$ $\quad e^x(x^2 + y^2) = C.$
8. $(2ax + by + g)\,dx + (2cy + bx + e)\,dy = 0.$

228. Integrating factor. When the equation,

$$Mdx + Ndy = 0,$$

is not exact, it may sometimes be made exact by multiplying it by a factor called an *integrating factor.*

Sometimes an integrating factor may be found by inspection, as in the examples below:

Ex. 1. Solve $y\,dx - x\,dy = 0.$ (1)

Equation (1) is not exact, but when multiplied by y^{-2}, it becomes

$$\frac{y\,dx - x\,dy}{y^2} = 0,$$

which is exact, and which has for its solution

$$x/y = C. \tag{2}$$

Multiplied by $1/xy$, (1) becomes

$$dx/x - dy/y = 0,$$

which is exact, and has for its solution

$$\log(x/y) = \log C. \tag{3}$$

Solution (2) is readily obtained from (3).

Multiplying (1) by x^{-2}, we obtain the solution

$$y/x = C_1.$$

Ex. 2. Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$ (1)

From (1), $y\,dx + x\,dy + xy^2\,dx - x^2y\,dy = 0,$

or $d(xy) + xy^2\,dx - x^2y\,dy = 0.$ (2)

Dividing (2) by x^2y^2, we obtain

$$\frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0.$$

$$\therefore -\frac{1}{xy} + \log\frac{x}{y} = \log C, \text{ or } x = Cye^{1/xy}.$$

229. Rules for finding the integrating factor of $Mdx + Ndy = 0.$ (a)

We give below the rules in four cases:

Rule I. When $Mx + Ny$ is not equal to zero, and (a) is homogeneous, $(Mx + Ny)^{-1}$ is an integrating factor of (a).

Ex. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ (1)

Equation (1) is homogeneous, and

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2.$$

Hence, $(x^2y^2)^{-1}$ is an integrating factor of (1).

Dividing (1) by x^2y^2, we obtain the exact equation

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0. \qquad (2)$$

Solving (2) by § 227, we obtain

$$x/y + \log(y^3/x^2) = C.$$

Rule II. When $Mx - Ny$ is not equal to zero, and (a) is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0, \qquad (b)$$

$(Mx - Ny)^{-1}$ is an integrating factor of (a).

Ex. Solve $(x^2y^2 + xy)\,y\,dx + (x^2y^2 - 1)\,x\,dy = 0.$ (1)

Equation (1) is of the form of (b), and

$$Mx - Ny = x^2y^2 + xy.$$

Hence, $(x^2y^2 + xy)^{-1}$ is an integrating factor of (1).

Divided by $x^2y^2 + xy$, (1) becomes

$$y\,dx + x\,dy = dy/y.$$

$$\therefore xy + \log C = \log y, \text{ or } y = Ce^{xy}.$$

RULE III. When $\frac{1}{N}\left(\frac{\partial M}{dy}-\frac{\partial N}{dx}\right)$ is a function of x alone, say fx, $e^{\int f(x)\,dx}$ is an integrating factor of (a).

Ex. Solve $(x^2+y^2+2x)\,dx+2y\,dy=0$. (1)

Here $$\frac{1}{N}\left(\frac{\partial M}{dy}-\frac{\partial N}{dx}\right)=\frac{1}{2y}(2y-0)=1=f(x).$$

Hence, $e^{\int f(x)\,dx}=e^{\int dx}=e^x$, the integrating factor of (1).

Multiplying (1) by e^x, we obtain the exact equation

$$e^x(x^2+y^2+2x)\,dx+2e^xy\,dy=0.$$

$$\therefore e^x(x^2+y^2)=C.$$ § 227, example 7

RULE IV. When $\frac{1}{M}\left(\frac{\partial N}{dx}-\frac{\partial M}{dy}\right)$ is a function of y alone, say fy, $e^{\int f(y)\,dy}$ is an integrating factor of (a).

230. Proof of rules in § 229. I. When $Mdx+Ndy$ is homogeneous,

$$\begin{aligned}Mdx+Ndy &\equiv \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)\right.\\ &\qquad\left.+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]\\ &\equiv \frac{1}{2}\left[(Mx+Ny)\,d\log(xy)\right.\\ &\qquad\left.+(Mx-Ny)\,d\log(x/y)\right]. \qquad (1)\end{aligned}$$

$$\begin{aligned}\therefore \frac{Mdx+Ndy}{Mx+Ny} &\equiv \frac{1}{2}d\log(xy)+\frac{1}{2}\frac{Mx-Ny}{Mx+Ny}d\log\frac{x}{y}\\ &\equiv \frac{1}{2}d\log(xy)+\frac{1}{2}f\left(\frac{x}{y}\right)\frac{d(x/y)}{x/y}. \qquad (2)\end{aligned}$$

$\frac{Mx-Ny}{Mx+Ny}$ is evidently equal to some function of x/y, when M and N are homogeneous.

The second member of (2) is an exact differential; hence, $(Mx+Ny)^{-1}$ is an integrating factor of (a).

Equation (2) fails when $Mx+Ny=0$. But in this case

$$M/N=-y/x.$$

Substituting this value of M/N in (a) of § 229, we obtain

$$\log (y/x)=\log C, \text{ or } y=Cx.$$

II. When $Mdx+Ndy$ is of the form

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy.$$

Dividing (1) by $Mx-Ny$, we obtain

$$\frac{Mdx+Ndy}{Mx-Ny}\equiv\frac{1}{2}\frac{Mx+Ny}{Mx-Ny}d(\log xy)+\frac{1}{2}d\left(\log\frac{x}{y}\right) \qquad (3)$$

$$\equiv\frac{1}{2}\frac{f_1(xy)\,xy+f_2(xy)\,xy}{f_1(xy)\,xy-f_2(xy)\,xy}\cdot\frac{d\,(xy)}{xy}+\frac{1}{2}d\left(\log\frac{x}{y}\right)$$

$$=F(xy)\frac{d\,(xy)}{xy}+\frac{1}{2}d\left(\log\frac{x}{y}\right). \qquad (4)$$

The second member of (4) is an exact differential; hence, $(Mx-Ny)^{-1}$ is an integrating factor of (a).

III. When $\frac{1}{N}\left(\frac{\partial M}{dy}-\frac{\partial N}{dx}\right)=fx.$

Multiplying (a) by $e^{\int f(x)\,dx}$, we obtain

$$e^{\int f(x)\,dx}Mdx+e^{\int f(x)\,dx}Ndy=0,$$

which is exact, for by differentiation we find that

$$\frac{\partial}{dy}\left(e^{\int f(x)\,dx}M\right)\equiv\frac{\partial}{dx}\left(e^{\int f(x)\,dx}N\right). \qquad \S\ 227$$

In like manner Rule IV is proved.

EXAMPLES.

1. $(x^2 + 2xy - y^2)\,dx = (x^2 - 2xy - y^2)\,dy.$ $\qquad x^2 + y^2 = C(x + y).$

2. $(x^2y^2 + xy^3)\,dx = (x^3y + x^2y^2)\,dy,$ $\qquad y = Cx.$

3. $\dfrac{dx}{x} + \dfrac{dy}{y} + 2\left(\dfrac{dx}{y} - \dfrac{dy}{x}\right) = 0.$ $\qquad x^2 - y^2 + xy = C.$

4. $x^3dx + (3x^2y + 2y^3)\,dy = 0.$ $\qquad x^2 + 2y^2 = C\sqrt{x^2 + y^2}.$

5. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$ $\qquad x = Cye^{\frac{1}{xy}}.$

6. $(\sqrt{xy} - 1)\,x\,dy = (\sqrt{xy} + 1)\,y\,dx.$ $\qquad 2/\sqrt{xy} = \log(Cx/y).$

7. $(y + y\sqrt{xy})\,dx + (x + x\sqrt{xy})\,dy = 0.$ $\qquad xy = C.$

8. $e^{xy}(x^2y^2 + xy)(x\,dy + y\,dx) + y\,dx - x\,dy = 0.$ $\qquad xye^{xy} = \log(Cy/x).$

9. $(3x^2 - y^2)\,dy = 2xy\,dx.$ $\qquad x^2 - y^2 = Cy^3.$

10. $2xy\,dy = (x^2 + y^2)\,dx.$ $\qquad y^2 - x^2 = Cx.$

231. Linear equations of the first order. A *linear* differential equation is one in which the dependent variable and its differentials appear only in the first degree.

The form of the linear equation of the first order is

$$dy + Py\,dx = Q\,dx, \tag{1}$$

where P and Q are functions of x or are constants.

The solution of $\quad dy + Py\,dx = 0$

is
$$\log y + \log e^{\int P\,dx} = \log C,$$

or
$$ye^{\int P\,dx} = C. \tag{2}$$

Differentiating (2), we obtain

$$e^{\int P\,dx}(dy + Py\,dx) = 0, \tag{3}$$

which shows that $e^{\int P\,dx}$ is an integrating factor of (1).

Multiplying (1) by $e^{\int P dx}$ and integrating the result, we obtain

$$ye^{\int P dx} = \int e^{\int P dx} Q\, dx. \tag{4}$$

Equality (4) may be used as a formula for solving any linear equation in the general form (1).

EXAMPLES.

1. $(1 + x^2)\, dy - yx\, dx = a\, dx.$ (1)

Putting (1) in the general form, we obtain

$$dy - \frac{x}{1 + x^2} y\, dx = \frac{a}{1 + x^2} dx.$$

Hence, $$\int P\, dx = -\int \frac{x\, dx}{1 + x^2} = \log (1 + x^2)^{-1/2}.$$

$$\therefore e^{\int P\, dx} = e^{\log (1 + x^2)^{-1/2}} = (1 + x^2)^{-1/2},$$

and $$\int e^{\int P\, dx} Q\, dx = \int \frac{a\, dx}{(1 + x^2)^{3/2}} = \frac{ax}{(1 + x^2)^{1/2}} + C.$$

Substituting these values in formula (4), we obtain

$$y = ax + C\sqrt{1 + x^2}.$$

2. $x\dfrac{dy}{dx} - ay = x + 1.$ $\qquad y = \dfrac{x}{1 - a} - \dfrac{1}{a} + Cx^a.$

3. $dy + y\, dx = e^{-x} dx.$ $\qquad y = (x + C)\, e^{-x}.$

4. $x (1 - x^2)\, dy + (2x^2 - 1)\, y\, dx = ax^3 dx.$ $\qquad y = ax + Cx\sqrt{1 - x^2}.$

5. $\cos x \cdot dy + y \sin x \cdot dx = dx.$ $\qquad y = \sin x + C \cos x.$

6. $(x^2 + 1)\, dy + 2xy\, dx = 4x^2 dx.$ $\qquad 3 (x^2 + 1)\, y = 4x^3 + C.$

7. $dy + y \cos x \cdot dx = (1/2) \sin 2x \cdot dx.$ $\qquad y = \sin x - 1 + Ce^{-\sin x}.$

232. Equations reducible to the linear form. Of such equations the most important are those of the form

$$dy/dx + Py = Qy^n, \tag{1}$$

where P and Q are functions of x or constants.

Assume $$z = y^{1-n};$$

then $$y = z^{\frac{1}{1-n}}, \qquad y^n = z^{\frac{n}{1-n}},$$

$$dy = \frac{1}{1-n} z^{\frac{n}{1-n}} dz.$$

Substituting these values in (1), we obtain

$$dz/dx + (1-n)Pz = (1-n)Q,$$

which is linear in z.

EXAMPLES.

1. $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2y^6.$ (1)

Assume $$z = y^{-5};$$

then $$y = z^{-1/5}, \quad dy = -(1/5)z^{-6/5}dz, \quad y^6 = z^{-6/5}.$$

Substituting these values in (1), we obtain

$$dz/dx - 5z/x = -5x^2. \tag{2}$$

Solving (2) by § 231 and putting y^{-5} for z, we have

$$y^{-5} = Cx^5 + 5x^3/2.$$

2. $(1-x^2)dy = (axy^2 + xy)dx.$ $\qquad y = (c\sqrt{1-x^2} - a)^{-1}.$

3. $3y^2dy = (x + 1 + ay^3)dx.$ $\qquad y^3 = Ce^{ax} - \dfrac{x+1}{a} - \dfrac{1}{a^2}.$

4. $dy = (x^3y^3 - xy)dx.$ $\qquad y^{-2} = x^2 + 1 + Ce^{x^2}.$

5. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x^2y^{4/3}.$ $\qquad 7y^{-1/3} = Cx^{2/3} - 3x^3.$

6. $(1-x^2)dy + xy\,dx = xy^{1/2}dx.$ $\qquad y^{1/2} = C(1-x^2)^{1/4} + 1.$

233. Equations of the first order and nth degree. We shall consider only such equations of the first order and nth degree as can be resolved into n equivalent rational equations of the first degree and of such types as have been solved in this chapter.

The method will be made clear by the following example, where $p = dy/dx$.

Ex. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ (1)

Equation (1) is equivalent to the equation

$$p(p + 2x)(p - y^2) = 0,$$

which is equivalent to the three equations

$$p = 0, \ p + 2x = 0, \ p - y^2 = 0. \tag{2}$$

Solving each of the equations (2), we obtain

$$y = C, \ y + x^2 = C, \ xy + Cy + 1 = 0, \tag{3}$$

where we have regarded all the constants of integration as equal.

Combining the three equations in (3) into one, we obtain

$$(y - C)(y + x^2 - C)(xy + Cy + 1) = 0. \tag{4}$$

Equation (4), or the three equations in (3), is the solution of (1).

EXAMPLES.

1. $p^2 - 7p + 12 = 0.$ $\quad (y - 4x - C)(y - 3x - C) = 0.$

2. $p^2 - ax^3 = 0.$ $\quad 25(y + C)^2 = 4ax^5.$

3. $p^2 - 5p + 6 = 0.$ $\quad (y - 2x - C)(y - 3x - C) = 0.$

4. $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0.$
$\quad (y - C)(x + y - C)(xy + x^2 + y^2 - C) = 0.$

5. $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0.$
$\quad (x^2 + 2y^2 - C)(x^3 + y^2 - C) = 0.$

234. Equations of orders above the first. We shall illustrate by examples the method of solving four special forms of such equations.

I. Equations of the form $d^ny/dx^n = fx$. (a)

Ex. Solve $d^3y/dx^3 = 5bx^2.$ (1)

Multiplying (1) by dx and integrating, we obtain

$$d^2y/dx^2 = (5/3)bx^3 + C_1. \tag{2}$$

Multiplying (2) by dx and integrating, we obtain

$$dy/dx = (5/12)bx^4 + C_1x + C_2. \tag{3}$$

$$\therefore y = bx^5/12 + C_1x^2/2 + C_2x + C_3.$$

II. Equations of the form $d^2y/dx^2 = fy$. (b)

Ex. Solve $d^2y/dx^2 + a^2y = 0$. (1)

Multiplying (1) by $2\,dy$, we obtain

$$2\frac{dy}{dx}d\frac{dy}{dx} = -2a^2y\,dy.$$

$$\therefore (dy/dx)^2 = -a^2y^2 + C_1 = a^2(c_1^2 - y^2), \tag{2}$$

where $C_1 = a^2c_1^2$.

From (2), $$dy/\sqrt{c_1^2 - y^2} = a\,dx.$$

$$\therefore \sin^{-1}(y/c_1) = ax + C_2,$$

or $$y = c_1 \sin(ax + C_2).$$

III. Equations of the form $f\left(\dfrac{d^ny}{dx^n}, \cdots, \dfrac{dy}{dx}, x\right) = 0$; (c)

that is, equations of the nth order not containing y directly.

Put $$p = \frac{dy}{dx};\ \text{then}\ \frac{d^2y}{dx^2} = \frac{dp}{dx}, \cdots, \frac{d^ny}{dx^n} = \frac{d^{n-1}p}{dx^{n-1}}.$$

Substituting these values in (c), we obtain

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \cdots, \frac{dp}{dx}, p, x\right) = 0, \tag{1}$$

which is an equation of the $(n-1)$th order between p and x.

Ex. Solve $d^2y/dx^2 = a^2 + b^2(dy/dx)^2$. (1)

Putting $p = dy/dx$, (1) becomes

$$dp/dx = a^2 + b^2p^2.$$

$$\therefore \tan^{-1}(bp/a) = ab(x + C_1),$$

or $$bp = a\tan[ab(x + C_1)].$$

$$\therefore b^2dy = \tan[ab(x + C_1)]\,ab\,dx.$$

$$\therefore b^2y = \log\sec[ab(x + C_1)] + C_2.$$

IV. Equations of the form $f\left(\dfrac{d^ny}{dx^n}, \cdots, \dfrac{dy}{dx}, y\right) = 0$. (d)

Put $$p = \frac{dy}{dx};\ \text{then}\ \frac{d^2y}{dx^2} = p\frac{dp}{dy},\ \frac{d^3y}{dx^3} = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \text{ etc.}$$

Substituting these values in (d), we obtain an equation of the $(n-1)$th order between p and y.

Ex. Solve $d^2y/dx^2 + a(dy/dx)^2 = 0$. (1)

Putting $p = dy/dx$, (1) becomes

$$\frac{dp}{dy} + ap = 0, \qquad \text{or } \frac{dp}{p} = -a\,dy.$$

$$\therefore p = C_1e^{-ay}, \quad \text{or } e^{ay}\,dy = C_1\,dx.$$

$$\therefore e^{ay} = C_1ax + C_2.$$

EXAMPLES.

Solve each of the following equations :

1. $d^3y = xe^x dx^3$. $\quad y = xe^x - 3e^x + C_1x^2 + C_2x + C_3$.

2. $d^4y = x^3 dx^4$.

3. $x\,d^3y = 2\,dx^3$. $\quad y = x^2 \log x + c_1x^2 + c_2x + c_3$,
where $c_1 = C_1/2 - 3/2$.

4. $d^2y = a^2y\,dx^2$. $\quad ax = \log(y + \sqrt{y^2 + c_1}) + c_2$,
where $a^2c_1 = C_1$.

5. $y^3\,d^2y = a\,dx^2$. $\quad c_1y^2 = c_1^2(x + c_2)^2 + a$.

6. $\sqrt{ay}\,d^2y = dx^2$. $\quad 3x = 2a^{1/4}(y^{1/2} - 2c_1)(y^{1/2} + c_1)^{1/2} + c_2$.

7. $x\,d^2y/dx^2 + dy/dx = 0$. $\quad y = c_1 \log x + c_2$.

8. $a^2(d^2y/dx^2)^2 = 1 + (dy/dx)^2$.
$\quad 2a^{-1}y = c_1e^{x/a} + c_1^{-1}e^{-x/a} + c_2$.

9. $(1 + x^2) \cdot d^2y/dx^2 + 1 + (dy/dx)^2 = 0$.
$\quad y = c_1x + (c_1^2 + 1)\log(c_1 - x) + c_2$.

10. $(1 - x^2) \cdot d^2y/dx^2 - x \cdot dy/dx = 2$.
$\quad y = c_1 \sin^{-1}x + (\sin^{-1}x)^2 + c_2$.

11. $y \cdot d^2y/dx^2 + (dy/dx)^2 = 1$. $\quad y^2 = x^2 + c_1x + c_2$.

12. The acceleration of a body moving toward a centre of attraction, C, varies directly as its distance from that centre ; determine the velocity and the time.

Let $a =$ the acceleration at a unit's distance from C;

$x =$ the *varying*, and c the *initial*, distance of the body from C; then $xa =$ the acceleration at the distance x.

Here $s = c - x$; $\therefore v = ds/dt = -dx/dt$; (1)

$$\therefore xa = d^2s/dt^2 = -d^2x/dt^2. \qquad (2)$$

Since $v = -dx/dt = 0$ when $x = c$, by integrating (2) we have

$$(dx/dt)^2 = ac^2 - ax^2.$$

$$\therefore v = -dx/dt = \sqrt{a(c^2 - x^2)}. \qquad (3)$$

Since $t = 0$ when $x = c$, from (3) we have

$$t = a^{-1/2}\cos^{-1}(x/c). \qquad (4)$$

Putting $x = 0$ in (3) and (4), we obtain

$v = c\sqrt{a}$, the velocity at the centre of force, C.

and $t = (1/2)\pi a^{-1/2}, (3/2)\pi a^{-1/2}, (5/2)\pi a^{-1/2}, \cdots$. (5)

Hence the motion is periodic, the time-period being $\pi/\sqrt{a}$, which is entirely independent of the initial distance.

The acceleration due to gravity at the earth's surface is 32.17 feet per second, and below the surface it varies as the distance from the centre. Hence, a particular case of the periodic motion considered above would be that of a body which could pass freely through the earth. Such a body would vibrate through the centre from surface to surface. Calling the diameter of the earth 20919360 feet, we would have in this case

$$a = 32.17/20919360;$$

$$\therefore \text{period} = \pi a^{-1/2} = 3.1416\sqrt{20919360/32.17} \text{ sec.}$$

$$= 42 \text{ min. } 13.4 \text{ sec.}$$

13. Assuming that the acceleration of a falling body above the surface of the earth varies inversely as the square of its distance from the earth's centre, find the velocity and time.

Let $x =$ the *varying*, and c the *initial*, distance of the body from the earth's centre;

$r =$ the radius of the earth;

$g =$ the acceleration due to gravity at its surface;

$\alpha =$ the acceleration due to gravity at the distance x.

Here $s = c - x$, and from the law of fall

$$\alpha : g = r^2 : x^2; \text{ or } \alpha = gr^2/x^2.$$

$$\therefore -d^2x/dt^2 = gr^2/x^2. \qquad (1)$$

Since $v = 0$ when $x = c$, from (1) we have

$$v = -\frac{dx}{dt} = (2\,gr^2)^{1/2}\left(\frac{1}{x} - \frac{1}{c}\right)^{1/2}. \tag{2}$$

Since $t = 0$ when $x = c$, from (2) we obtain

$$t = \left(\frac{c}{2\,gr^2}\right)^{1/2}\int_c^x \frac{-x\,dx}{\sqrt{cx - x^2}}$$

$$= \left(\frac{c}{2\,gr^2}\right)^{1/2}\left[(cx - x^2)^{1/2} - \frac{c}{2}\left(\text{vers}^{-1}\frac{2x}{c} + \pi\right)\right].$$

14. Assuming that r, the radius of the earth, is 3962 miles; that the sun is 24,000 r distant from the earth; and that the moon is 60 r distant; find the time that it would take a body to fall from the moon to the earth, and the velocity, at the earth's surface, of a body falling from the sun. The attraction of the moon and the sun, and the resistance of any medium, are not to be considered.

15. A body falls in the air by the force of gravity, the resistance of the air varying as the square of the velocity; determine the velocity on the hypothesis that the force of gravity is constant.

Let b = the resistance when the velocity is unity;
and t = the time of falling through the distance s.
Then $b\,(ds/dt)^2$ = the resistance of the air for any velocity;
and g = the acceleration downward due to gravity alone.
Hence, $g - b\,(ds/dt)^2$ = the actual acceleration downward;
that is, $d^2s/dt^2 = g - b\,(ds/dt)^2$. (1)

Integrating (1) and solving for ds/dt, we obtain

$$v = \frac{ds}{dt} = \sqrt{\frac{g}{b}}\,\frac{e^{2t\sqrt{bg}} - 1}{e^{2t\sqrt{bg}} + 1}.$$

As t increases, v rapidly approaches the constant value $\sqrt{g/b}$.

16. A body is projected with a velocity, v_0, into a medium which resists as the square of the velocity; determine the velocity and the distance after t seconds.

Let b = the resistance of the medium when the velocity is unity;
then $b\,(ds/dt)^2$ = the resistance for any velocity.
Hence, $d^2s/dt^2 = -b\,(ds/dt)^2$. (1)

Integrating (1) and solving for ds/dt, we obtain

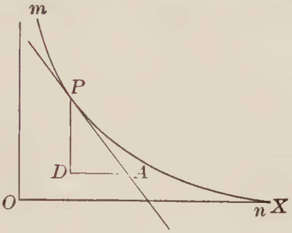
$$v = ds/dt = v_0/e^{bs}. \tag{2}$$

Integrating (2) and solving for s, we obtain

$$s = \log(bv_0t + 1)/b.$$

The velocity decreases rapidly and $\doteq 0$ when $s = \infty$.

17. A body slides without friction down any curve, mn. The acceleration caused by gravity at any point, P, is $g \cos DPA$, PA being a tangent. Find the velocity of the body.



Let $PA = ds$; then $-PD = dy$.

$$\therefore\ d^2s/dt^2 = g \cos DPA$$

$$= -g \cdot dy/ds. \qquad (1)$$

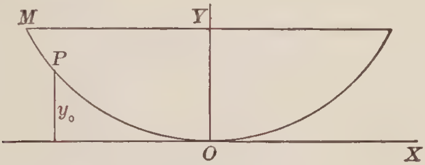
Let y_0 be the ordinate of the starting point on the curve; then $v = 0$ when $y = y_0$.

Integrating (1), we obtain

$$v = ds/dt = \sqrt{2g(y_0 - y)}. \qquad (2)$$

From (2) it follows that if a body falls from the line $y = y_0$ to the line $y = b$, the velocity acquired is the same for all curves of descent.

18. A body falls from the point P along the arc of a cycloid, PO; find the time of descent.



From (2) of example 17 we have

$$v = ds/dt = \sqrt{2g(y_0 - y)}. \qquad (1)$$

The equation of the cycloid referred to OX and OY is

$$x = r \text{ vers}^{-1}(y/r) + \sqrt{2ry - y^2}.$$

Hence, $\quad ds = -\sqrt{2r/y}\,dy. \qquad (2)$

Eliminating ds between (1) and (2) and integrating, we obtain

$$t = \sqrt{r/g}\,[\pi - \text{vers}^{-1}(2y/y_0)].$$

$$\therefore\ t = \pi\sqrt{r/g}, \text{ when } y = 0.$$

Hence, if a pendulum swings in the arc of a cycloid, the time required for one oscillation is $2\pi\sqrt{r/g}$.

The time of an oscillation being independent of the length of the arc, the cycloidal pendulum is isochronal.

APPENDIX.

FORMULAS FOR REFERENCE.

Standard Forms.

1. $\int u^n\,du = \frac{u^{n+1}}{n+1}.$ 3. $\int a^u\,du = \frac{a^u}{\log a}.$

2. $\int \frac{du}{u} = \log u.$ 4. $\int e^u\,du = e^u.$

5. $\int \sin u\,du = -\cos u,$ or $\operatorname{vers} u.$

6. $\int \cos u\,du = \sin u,$ or $-\operatorname{covers} u.$

7. $\int \sec^2 u\,du = \tan u.$ 11. $\int \tan u\,du = \log \sec u.$

8. $\int \csc^2 u\,du = -\cot u.$ 12. $\int \cot u\,du = \log \sin u.$

9. $\int \sec u \tan u\,du = \sec u.$ 13. $\int \csc u\,du = \log \tan \frac{u}{2}.$

10. $\int \csc u \cot u\,du = -\csc u.$ 14. $\int \sec u\,du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right).$

15. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a},$ or $-\frac{1}{a}\cot^{-1}\frac{u}{a}.$

16. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2\,a} \log \frac{u - a}{u + a}, \text{ or } \frac{1}{2\,a} \log \frac{a - u}{a + u}.$$

17. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}, \text{ or } -\cos^{-1} \frac{u}{a}.$$

18. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}).$$

19. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}, \text{ or } -\frac{1}{a} \csc^{-1} \frac{u}{a}.$$

20. $$\int \frac{du}{\sqrt{2\,au - u^2}} = \text{vers}^{-1} \frac{u}{a}, \text{ or } -\text{covers}^{-1} \frac{u}{a}.$$

Elementary Principles and Formulas.

21. $$\int \phi(u)\,du = fu + C, \text{ when } dfu = \phi(u)\,du.$$

22. $$\int (du + dy + dz) = \int du + \int dy + \int dz.$$

23. $$\int a\,du = a \int du. \qquad \int 0 = C.$$

24. $$\int u\,dv = uv - \int v\,du.$$

Forms Involving $a + bu$.

25. $$\int \frac{u\,du}{a + bu} = \frac{1}{b^2} \left[a + bu - a \log (a + bu) \right].$$

26. $$\int \frac{u\,du}{(a + bu)^2} = \frac{1}{b^2} \left[\log (a + bu) + \frac{a}{a + bu} \right].$$

27. $$\int \frac{u^2\,du}{a + bu} = \frac{1}{b^3} \left[\frac{(a + bu)^2}{2} - 2\,a\,(a + bu) + a^2 \log (a + bu) \right].$$

28. $$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[a + bu - 2\,a\log(a+bu) - \frac{a^2}{a+bu}\right].$$

29. $$\int \frac{du}{u(a+bu)} = -\frac{1}{a}\log\frac{a+bu}{u}.$$

30. $$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\log\frac{a+bu}{u}.$$

31. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\log\frac{a+bu}{u}.$$

Forms Involving $a + bu^2$.

32. $$\int \frac{du}{a+bu^2} = \frac{1}{\sqrt{ab}}\tan^{-1} u\sqrt{\frac{b}{a}}, \text{ when } a > 0 \text{ and } b > 0;$$

33. $$= \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}+u\sqrt{-b}}{\sqrt{a}-u\sqrt{-b}}, \text{ when } a > 0 \text{ and } b < 0.$$

34. $$\int \frac{du}{(a+bu^2)^2} = \frac{u}{2\,a(a+bu^2)} + \frac{1}{2\,a}\int\frac{du}{a+bu^2}.$$

35. $$\int \frac{du}{(a+bu^2)^{r+1}} = \frac{1}{2\,ra}\,\frac{u}{(a+bu^2)^r} + \frac{2\,r-1}{2\,ra}\int\frac{du}{(a+bu^2)^r}.$$

36. $$\int \frac{u^2\,du}{a+bu^2} = \frac{u}{b} - \frac{a}{b}\int\frac{du}{a+bu^2}.$$

37. $$\int \frac{u^2\,du}{(a+bu^2)^{r+1}} = \frac{-u}{2\,rb(a+bu^2)^r} + \frac{1}{2\,rb}\int\frac{du}{(a+bu^2)^r}.$$

38. $$\int \frac{du}{u(a+bu^2)} = \frac{1}{2\,a}\log\frac{u^2}{a+bu^2}.$$

39. $$\int \frac{du}{u^2(a+bu^2)} = -\frac{1}{au} - \frac{b}{a}\int\frac{du}{a+bu^2}.$$

40. $$\int \frac{du}{u^2(a+bu^2)^{r+1}} = \frac{1}{a}\int\frac{du}{u^2(a+bu^2)^r} - \frac{b}{a}\int\frac{du}{(a+bu^2)^{r+1}}.$$

Forms Involving $a + bu^n$.

41. $$\int u^m (a + bu^n)^p \, du$$
$$= \frac{u^{m-n+1}(a + bu^n)^{p+1}}{b(np + m + 1)} - \frac{a(m - n + 1)}{b(np + m + 1)} \int u^{m-n}(a + bu^n)^p \, du;$$

42. or $$\frac{u^{m+1}(a + bu^n)^p}{np + m + 1} + \frac{anp}{np + m + 1} \int u^m (a + bu^n)^{p-1} \, du;$$

43. or $$\frac{u^{m+1}(a + bu^n)^{p+1}}{a(m + 1)} - \frac{b(np + m + n + 1)}{a(m + 1)} \int u^{m+n}(a + bu^n)^p \, du;$$

44. or $$-\frac{u^{m+1}(a + bu^n)^{p+1}}{an(p + 1)} + \frac{np + m + n + 1}{an(p + 1)} \int u^m (a + bu^n)^{p+1} \, du.$$

Forms Involving $au^2 + bu + c$.

45. $$\int \frac{du}{au^2 + bu + c} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2au + b}{\sqrt{4ac - b^2}};$$

46. $$= \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{2au + b - \sqrt{b^2 - 4ac}}{2au + b + \sqrt{b^2 - 4ac}}.$$

47. $$\int \frac{u\,du}{au^2 + bu + c} = \frac{1}{2a} \log(au^2 + bu + c) - \frac{b}{2a} \int \frac{du}{au^2 + bu + c}.$$

Forms Involving $\sqrt{a + bu}$.

48. $$\int u\sqrt{a + bu}\, du = -\frac{2(2a - 3bu)(a + bu)^{3/2}}{15b^2}.$$

49. $$\int u^2\sqrt{a + bu}\, du = \frac{2(8a^2 - 12abu + 15b^2u^2)(a + bu)^{3/2}}{105b^3}.$$

50. $$\int \frac{u\,du}{\sqrt{a + bu}} = -\frac{2(2a - bu)\sqrt{a + bu}}{3b^2}.$$

51. $$\int \frac{u^n\,du}{\sqrt{a + bu}} = \frac{2u^n\sqrt{a + bu}}{(2n + 1)b} - \frac{2na}{(2n + 1)b} \int \frac{u^{n-1}\,du}{\sqrt{a + bu}}.$$

52. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}, \text{ when } a > 0;$$

53. $$= \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}}, \text{ when } a < 0.$$

54. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{(n-1)\,au^{n-1}} - \frac{(2n-3)\,b}{(2n-2)\,a} \int \frac{du}{u^{n-1}\sqrt{a+bu}}.$$

55. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}.$$

Forms Involving $\sqrt{a^2-u^2}$.

56. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}.$$

57. $$\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a} \log \frac{u}{a+\sqrt{a^2-u^2}}.$$

58. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u}.$$

59. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$$

60. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a}.$$

61. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a \log \frac{a+\sqrt{a^2-u^2}}{u}.$$

62. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a}.$$

63. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$$

64. $$\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}}.$$

65. $\int (a^2 - u^2)^{3/2}\,du = \frac{u}{8}(5\,a^2 - 2\,u^2)\sqrt{a^2 - u^2} + \frac{3\,a^4}{8}\sin^{-1}\frac{u}{a}.$

66. $\int \frac{u^2\,du}{(a^2 - u^2)^{3/2}} = \frac{u}{\sqrt{a^2 - u^2}} - \sin^{-1}\frac{u}{a}.$

Forms Involving $\sqrt{u^2 \pm a^2}$.

67. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log(u + \sqrt{u^2 \pm a^2}).$

68. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}.$

69. $\int \frac{du}{u\sqrt{u^2 + a^2}} = \frac{1}{a}\log\frac{u}{a + \sqrt{u^2 + a^2}}.$

70. $\int \frac{du}{u^2\sqrt{u^2 \pm a^2}} = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2 u}.$

71. $\int \frac{du}{u^3\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{2\,a^2u^2} + \frac{1}{2\,a^3}\sec^{-1}\frac{u}{a}.$

72. $\int \frac{du}{u^3\sqrt{u^2 + a^2}} = -\frac{\sqrt{u^2 + a^2}}{2\,a^2u^2} + \frac{1}{2\,a^3}\log\frac{a + \sqrt{u^2 + a^2}}{u}.$

73. $\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\log(u + \sqrt{u^2 \pm a^2}).$

74. $\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{8}(2\,u^2 \pm a^2)\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\log(u + \sqrt{u^2 \pm a^2}).$

75. $\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{u}.$

76. $\int \frac{\sqrt{u^2 + a^2}}{u}\,du = \sqrt{u^2 + a^2} - a\log\frac{a + \sqrt{u^2 + a^2}}{u}.$

77. $\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\,du = -\frac{\sqrt{u^2 \pm a^2}}{u} + \log(u + \sqrt{u^2 \pm a^2}).$

78. $$\int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2}\log\left(u + \sqrt{u^2 \pm a^2}\right).$$

79. $$\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{\pm\, u}{a^2\sqrt{u^2 \pm a^2}}.$$

80. $$\int \frac{u^2\,du}{(u^2 \pm a^2)^{3/2}} = \frac{-\,u}{\sqrt{u^2 \pm a^2}} + \log\left(u + \sqrt{u^2 \pm a^2}\right).$$

81. $$\int (u^2 \pm a^2)^{3/2}\,du$$
$$= \frac{u}{8}(2\,u^2 \pm 5\,a^2)\sqrt{u^2 \pm a^2} + \frac{3\,a^4}{8}\log\left(u + \sqrt{u^2 \pm a^2}\right).$$

Forms Involving $\sqrt{2\,au - u^2}$.

82. $$\int \frac{du}{\sqrt{2\,au - u^2}} = \text{vers}^{-1}\frac{u}{a}.$$

83. $$\int \frac{u^m\,du}{\sqrt{2\,au - u^2}} = -\frac{u^{m-1}\sqrt{2\,au - u^2}}{m} + \frac{(2\,m - 1)\,a}{m}\int \frac{u^{m-1}\,du}{\sqrt{2\,au - u^2}}.$$

84. $$\int \frac{du}{u^m\sqrt{2\,au - u^2}}$$
$$= -\frac{\sqrt{2\,au - u^2}}{(2\,m - 1)\,au^m} + \frac{m - 1}{(2\,m - 1)\,a}\int \frac{du}{u^{m-1}\sqrt{2\,au - u^2}}.$$

85. $$\int \sqrt{2\,au - u^2}\,du = \frac{u - a}{2}\sqrt{2\,au - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u - a}{a}.$$

86. $$\int u^m\sqrt{2\,au - u^2}\,du$$
$$= -\frac{u^{m-1}(2\,au - u^2)^{3/2}}{m + 2} + \frac{(2\,m + 1)\,a}{m + 2}\int u^{m-1}\sqrt{2\,au - u^2}\,du.$$

87. $$\int \frac{\sqrt{2\,au - u^2}\,du}{u^m}$$
$$= -\frac{(2\,au - u^2)^{3/2}}{(2\,m - 3)\,au^m} + \frac{m - 3}{(2\,m - 3)\,a}\int \frac{\sqrt{2\,au - u^2}\,du}{u^{m-1}}.$$

Forms Involving $\sqrt{\pm au^2 + bu + c}$, where $a > 0$.

88. $$\int \frac{du}{\sqrt{au^2 + bu + c}} = \frac{1}{\sqrt{a}} \log (2\,au + b + 2\sqrt{a}\sqrt{au^2 + bu + c}).$$

89. $$\int \sqrt{au^2 + bu + c}\,du$$
$$= \frac{2\,au + b}{4\,a}\sqrt{au^2 + bu + c} - \frac{b^2 - 4\,ac}{8\,a}\int \frac{du}{\sqrt{au^2 + bu + c}}.$$

90. $$\int \frac{du}{\sqrt{-au^2 + bu + c}} = \frac{1}{\sqrt{a}} \sin^{-1} \frac{2\,au - b}{\sqrt{b^2 + 4\,ac}}.$$

91. $$\int \sqrt{-au^2 + bu + c}\,du$$
$$= \frac{2\,au - b}{4\,a}\sqrt{-au^2 + bu + c} + \frac{b^2 + 4\,ac}{8\,a}\int \frac{du}{\sqrt{-au^2 + bu + c}}.$$

92. $$\int \frac{u\,du}{\sqrt{\pm au^2 + bu + c}}$$
$$= \frac{\sqrt{\pm au^2 + bu + c}}{\pm a} \mp \frac{b}{2\,a}\int \frac{du}{\sqrt{\pm au^2 + bu + c}}.$$

93. $$\int u\sqrt{\pm au^2 + bu + c}\,du$$
$$= \frac{(\pm au^2 + bu + c)^{3/2}}{3\,a} \mp \frac{b}{2\,a}\int \sqrt{\pm au^2 + bu + c}\,du.$$

Forms Involving Transcendental Functions.

94. $$\int \sin^2 u\,du = \frac{u}{2} - \frac{1}{4}\sin 2\,u.$$

95. $$\int \cos^2 u\,du = \frac{u}{2} + \frac{1}{4}\sin 2\,u.$$

96. $$\int \sin^2 u \cos^2 u\,du = \frac{1}{8}\left(u - \frac{1}{4}\sin 4\,u\right).$$

97. $$\int \sec u \csc u\,du = \int \frac{du}{\sin u \cos u} = \log \tan u.$$

98. $$\int \sec^2 u \csc^2 u\, du = \int \frac{du}{\sin^2 u \cos^2 u} = \tan u - \cot u.$$

99. $$\int \sin^m u \cos^n u\, du$$
$$= -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u \cos^n u\, du;$$

100. $$= \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u \cos^{n-2} u\, du.$$

101. $$\int \sin^m u\, du = -\frac{\sin^{m-1} u \cos u}{m} + \frac{m-1}{m}\int \sin^{m-2} u\, du.$$

102. $$\int \cos^n u\, du = \frac{\sin u \cos^{n-1} u}{n} + \frac{n-1}{n}\int \cos^{n-1} u\, du.$$

103. $$\int \frac{\sin^m u}{\cos^n u} du = \frac{\sin^{m+1} u}{(n-1)\cos^{n-1} u} + \frac{n-m-2}{n-1}\int \frac{\sin^m u\, du}{\cos^{n-2} u}.$$

104. $$\int \frac{\cos^n u}{\sin^m u} du = -\frac{\cos^{n+1} u}{(m-1)\sin^{m-1} u} + \frac{m-n-2}{m-1}\int \frac{\cos^n u\, du}{\sin^{m-2} u}.$$

105. $$\int \frac{du}{\sin^m u} = -\frac{\cos u}{(m-1)\sin^{m-1} u} + \frac{m-2}{m-1}\int \frac{du}{\sin^{m-2} u}.$$

106. $$\int \frac{du}{\cos^n u} = \frac{\sin u}{(n-1)\cos^{n-1} u} + \frac{n-2}{n-1}\int \frac{du}{\cos^{n-2} u}.$$

107. $$\int \tan^n u\, du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du.$$

108. $$\int \cot^n u\, du = -\frac{\cot^{n-1} u}{n-1} - \int \cot^{n-2} u\, du.$$

109. $$\int \frac{du}{a + b\cos u} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{u}{2}\right), \text{ if } a^2 > b^2;$$

110. $$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b-a}\tan\frac{u}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{u}{2} - \sqrt{b+a}}, \text{ if } a^2 < b^2.$$

111. $$\int u^m \sin u\,du = -u^m \cos u + m\int u^{m-1}\cos u\,du.$$

112. $$\int u^m \cos u\,du = u^m \sin u - m\int u^{m-1}\cos u\,du.$$

113. $$\int \frac{\sin u}{u}\,du = u - \frac{u^3}{3\cdot\underline{|3}} + \frac{u^5}{5\cdot\underline{|5}} - \frac{u^7}{7\cdot\underline{|7}} + \frac{u^9}{9\cdot\underline{|9}}\cdots.$$

114. $$\int \frac{\sin u}{u^m}\,du = -\frac{1}{m-1}\,\frac{\sin u}{u^{m-1}} + \frac{1}{m-1}\int\frac{\cos u}{u^{m-1}}\,du.$$

115. $$\int \frac{\cos u}{u}\,du = \log u - \frac{u^2}{2\cdot\underline{|2}} + \frac{u^4}{4\cdot\underline{|4}} - \frac{u^6}{6\cdot\underline{|6}} + \frac{u^8}{8\cdot\underline{|8}}\cdots.$$

116. $$\int \frac{\cos u}{u^m}\,du = -\frac{1}{m-1}\,\frac{\cos u}{u^{m-1}} - \frac{1}{m-1}\int\frac{\sin u}{u^{m-1}}\,du.$$

117. $$\int u\sin^{-1}u\,du = \frac{1}{4}[(2u^2-1)\sin^{-1}u + u\sqrt{1-u^2}].$$

118. $$\int u^n \sin^{-1}u\,du = \frac{u^{n+1}\sin^{-1}u}{n+1} - \frac{1}{n+1}\int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}.$$

119. $$\int u^n \cos^{-1}u\,du = \frac{u^{n+1}\cos^{-1}u}{n+1} + \frac{1}{n+1}\int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}.$$

120. $$\int u^n \tan^{-1}u\,du = \frac{u^{n+1}\tan^{-1}u}{n+1} - \frac{1}{n+1}\int\frac{u^{n+1}\,du}{1+u^2}.$$

121. $$\int u^n \log u\,du = u^{n+1}\left[\frac{\log u}{n+1} - \frac{1}{(n+1)^2}\right].$$

122. $$\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du.$$

123. $$\int \frac{e^{au}}{u^n}\,du = -\frac{1}{n-1}\cdot\frac{e^{au}}{u^{n-1}} + \frac{a}{n-1}\int\frac{e^{au}}{u^{n-1}}\,du.$$

124. $$\int e^{au}\log u\,du = \frac{e^{au}\log u}{a} - \frac{1}{a}\int\frac{e^{au}}{u}\,du.$$

125. $$\int e^{au}\sin nu\,du=\frac{e^{au}(a\sin nu-n\cos nu)}{a^2+n^2}.$$

126. $$\int e^{au}\cos nu\,du=\frac{e^{au}(a\cos nu+n\sin nu)}{a^2+n^2}.$$

Miscellaneous Forms.

127. $$\int\sqrt{\frac{a+u}{b+u}}\,du$$
$$=\sqrt{(a+u)(b+u)}+(a-b)\log\left(\sqrt{a+u}+\sqrt{b+u}\right).$$

To prove formula 127 let $b+u=z^2$.

128. $$\int\sqrt{\frac{a-u}{b+u}}\,du=\sqrt{(a-u)(b+u)}+(a+b)\sin^{-1}\sqrt{\frac{b+u}{b+a}}.$$

129. $$\int\frac{du}{\sqrt{(u-a)(b-u)}}=2\cot^{-1}\sqrt{\frac{b-u}{u-a}}=2\sin^{-1}\sqrt{\frac{u-a}{b-a}}.$$

130. $$\int\frac{du}{u\sqrt{u^n+a^2}}=\frac{1}{an}\log\frac{\sqrt{a^2+u^n}-a}{\sqrt{a^2+u^n}+a}.$$

131. $$\int\frac{du}{u\sqrt{u^n-a^2}}=\frac{2}{an}\sec^{-1}\frac{u^{\frac{n}{2}}}{a}.$$

SHORT COURSE IN THE CALCULUS.

To those who wish to give in Taylor's Calculus a short course including the fundamental principles, problems, methods, and applications of the Calculus, the following suggestions may prove helpful:

PART I.

Chapter I. Take it all thoroughly.

Chapter II. Omit exs. 28–34, pp. 18, 19; exs. 6, 8, 12, 13, 14, 25–30, pp. 19, 20; exs. 8–10, pp. 21, 22; exs. 19–23, pp. 28, 29; exs. 23–29, p. 32; exs. 18–23, pp. 35, 36; exs. 1–30, pp. 38, 39.

Chapter III. Omit all after p. 44 except § 74.

Chapter IV. Omit all after ex. 15, p. 60, except the definition in § 82.

Chapter V. Omit exs. 9–14, p. 67; exs. 8–11, p. 68; all the chapter after ex. 9, p. 69.

Chapter VI. Read the proof of Taylor's theorem with the class, but do not require its reproduction. Omit Cors. 2–4, p. 74; §§ 93, 96, 99; the proofs of convergency in §§ 94, 95, 97, 98; exs. 11–20, p. 81.

Chapter VII. Omit exs. 12–19 and 22, p. 87; exs. 14–16, p. 90; exs. 19–24, pp. 91, 92.

Chapter VIII. Omit § 110; exs. 8–11, p. 97; § 116 and examples; exs. 7–9, p. 102.

Chapter IX. Omit exs. 8–13, pp. 106, 107; exs. 1–8, p. 111.

Chapter X. Omit it all.

Chapter XI. Omit pp. 118–126.

Chapter XII. Omit exs. 7–9, p. 130; exs. 9–12, p. 132; § 149 and examples; § 153 and examples. Read curve tracing with the class so that the most important curves are made familiar.

PART II.

Chapter I. Omit exs. 39–48, p. 150; exs. 57–59, p. 151; exs. 22–24, p. 153; exs. 19–24, pp. 155–156.

Chapter II. Omit exs. 4–6, p. 160; § 168, p. 166.

Chapter III. Omit exs. 7–9, p. 169; exs. 6–8, p. 170; exs. 8–12, p. 172; exs. 3–6, p. 173.

Chapter IV. Omit exs. 6, 10, p. 175; exs. 7–9, p. 177; exs. 8–11, p. 180.

Chapter V. Omit exs. 17–26, p. 184; exs. 10, 14, 19–21, 25–31, pp. 188, 189.

Chapter VI. Omit exs. 11–15, p. 192; exs. 10–13, p. 194; exs. 5–7, p. 195; §§ 190, 191, and examples, pp. 196, 197; § 192, and examples, pp. 198, 199; exs. 5–9, p. 200; § 194 and examples, p. 201; exs. 7–9, p. 202.

Chapter VII. Omit exs. 5, 7, p. 204; § 198; exs. 3, 4, p. 207; § 200 and examples; exs. 7–13, p. 210; exs. 6–9, p. 211; exs. 5–8 and 13–17, pp. 214, 215; § 205 and examples.

Chapter VIII. Omit all after ex. 3, p. 221.

To gain an idea of limits and infinitesimals as used in the Calculus, read Chapter III in Part I and Chapter IX in Part II.

If a course still shorter than that outlined above is desired, omit all of Chapters VII–XII in Part I, all of Chapter VIII in Part II, and more examples in the other chapters.

[illegible]

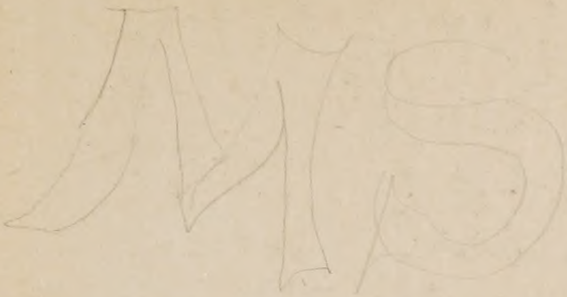
[illegible] $+ Pl -$ [illegible]

[illegible]

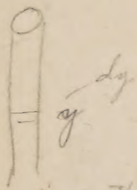
[illegible] $=$ [illegible]

[illegible]

McDonald G M



$$\frac{2x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$2000' - 5^{II}$